CSCI 104
Counting
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Tips

• Know some of the basic rules and formulas we provide
• But don't just look for a formula as your first step. STOP! Think through the problem, your intuitions, and what rules/formulas *may* seem useful
• Write some examples and then determine your approach
• Often you may need to break the problem up and apply the rules/formulas to the parts
Brute-Force Counting

• How many ways are there to roll a total of 6 with two standard dice?

• How many bit strings of length 4 do not have two consecutive 1s?
Brute-Force Counting

• How many ways are there to roll a total of 6 with two standard dice?
  ➢ 1-5, 2-4, 3-3, 4-2, 5-1 = 5

• How many bit strings of length 4 do not have two consecutive 1s?
FUNDAMENTAL COUNTING RULES
Key Ideas

• Fundamental Counting Rules
  – **Product, Division, Subtraction**, and **Addition** rules
  – These can be used to fully solve "simple" problems or as a technique to decompose a problem into or solve subproblems

• Permutations and Combinations
  – Counting subsets when **order** DOES or DOES NOT matter
  – Counting techniques differ when elements/options CAN or CANNOT be **replaced** (chosen again)

• Distributing elements to "boxes/bins"
  – Different techniques based on distinguishability and indistinguishability of both elements and the "boxes/bins"
PERMUTATIONS AND COMBINATIONS
Product Rule Motivation

• A company has 3 new employees: Aaron, Simone, and Jenna. There are 12 offices. How many different ways are there to assign offices?

A   S   J
The Product Rule

• A company has 3 new employees: Aaron, Simone, and Jenna. There are 12 offices. How many different ways are there to assign offices?
  – \(12 \times 11 \times 10 = 1320\)

• The **Product Rule** states that if a procedure can be broken up into a sequence of \(k\) tasks, and there are \(n_i\) ways to do the \(i\)-th task, then there are a total of

\[
\prod_{i=1}^{k} n_i
\]

ways to do the procedure,

or \(|A \times B| = |A| \cdot |B|\)

<table>
<thead>
<tr>
<th>{ABC}x{1,2}</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A1</td>
<td>A2</td>
</tr>
<tr>
<td>B</td>
<td>B1</td>
<td>B2</td>
</tr>
<tr>
<td>C</td>
<td>C1</td>
<td>C2</td>
</tr>
</tbody>
</table>
Replacement Or Non-Replacement

• Ex. 1: How many ways are there to arrange 10 people in a line?

• Ex. 2: If a license plate must have 3 letters followed by 3 digits, how many unique license plates are there?

• A helpful discriminator in solving these kinds of problems is to determine if the same option can be reused (aka "replaced/used multiple times") in subsequence selections (tasks)
  – Without replacement: Answer often involves __________
    • How many ways to arrange \{A,B,C,D\}? ___ ___ ___ ___ ___

  – With replacement: Answer often involves __________
    • How many unique strings exist use only characters \{A,B,C,D\}? ___ ___ ___ ___
Replacement Or Non-Replacement

Ex. 1: How many ways are there to arrange 10 people in a line?
- $10! = 3628800$

Ex. 2: If a license plate must have 3 letters followed by 3 digits, how many unique license plates are there?
- $26^3 \cdot 10^3 = 17576000$

A helpful discriminator in solving these kinds of problems is to determine if the same option can be reused (aka "replaced/used multiple times") in subsequence selections (tasks)
- Without replacement: Answer often involves **factorials**
  - How many ways to arrange \{A,B,C,D\}? ___ ___ ___ ___
  - $4!$
- With replacement: Answer often involves **exponents**
  - How many length 3 strings exist use only characters \{A,B,C,D\}? ___ ___ ___
  - $4^3$
Division Rule Motivation

• How many different ways are there to seat 4 people around a circular table, where two seatings are considered identical if each person has the same left neighbor in both seatings, and the same right neighbor in both seatings?

• How many ways are there to arrange 10 people in a line, when it is unspecified which end is the front?
The Division Rule

• How many different ways are there to seat 4 people around a circular table, where two seatings are considered identical if each person has the same left neighbor in both settings, and the same right neighbor in both settings?
  
  \[ \frac{4!}{4} = 6 \]

• The **Division Rule** states that there are \( \frac{n}{d} \) ways to do a task which can be done in one of \( n \) different ways, but for each specific way, it is identical to \( d \)-1 other ways.

• How many ways are there to arrange 10 people in a line, when it is unspecified which end is the front?
  
  \[ \frac{10!}{2} \]
r Permutations

• How many ways can we select 3 students from a group of 5 to stand in line for groceries?
  – Consider the use of the multiplication and division rule
r Permutations

- How many ways can we select 3 students from a group of 5 to stand in line for groceries?
  - Consider the use of the multiplication and division rule

- An $r$-permutation is an **ordered** arrangement of $r$ elements from a set of $n$, denoted $nPr$,
  
  $P(n,r)$, or $nP_r = \frac{n!}{(n-r)!}$
  
  - Note: Permuting all $n$ items in the set yields: $(nP_n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$
Combinations

• How many ways are there to select 3 representatives from 5 students to go to a club meeting

\[ \frac{5!}{2! \cdot 3!} \]

• An \( r \)-combination is an unordered arrangement of \( r \) elements from a set of size \( n \), is commonly spoken as "\( n \) choose \( r \)" and is denoted \( nC_r \), \( C(n,r) \) or \( \binom{n}{r} \).

\[ nC_r = \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!} \]
Permutation/Combination Examples

• How many permutations of the letters DIJKSTRA contain the substring IJK?
  – 6!

• How many different 5-card Poker hands can be dealt from a standard deck of 52 cards?
  – \( \binom{52}{5} = 2,598,960 \)

• How many different 47-card hands can be dealt?
  – \( \binom{52}{47} = 2,598,960 \)
Properties of Combinations

• Fact 1: \( \binom{n}{r} = \binom{n}{n-r} \)
  
  \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \) and \( \binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!(r)!} \)

• Fact 2: \( \sum_{k=0}^{n} \binom{n}{k} = 2^{n} \)
  
  – This sum enumerates all possible subsets and yields the power set (of which there are \( 2^{n} \) total subsets)
  
  – Ex: \{A, B, C\}
Binomial Theorem & Pascal's Identity

• Expand \((x + y)^3 = (x + y)(x + y)(x + y)\)
  \[x^3 + 3x^2y + 3xy^2 + y^3\]
  \[\binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3\]

• The Binomial Theorem is as follows:
  \[(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j\]

• What is the coefficient of \(x^{12}y^{13}\) in the expansion of \((x + y)^{25}\)?
  \[\binom{25}{13} = \binom{25}{12}\]
Pascal’s Triangle and Identity

Pascal’s Identity:
\[
\binom{n + 1}{k} = \binom{n}{k - 1} + \binom{n}{k}
\]
Proof of Pascal's Identity

• Prove: \( \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \)
• Let T be a set with n+1 elements. Let a be an element in T, and let S = T – \{a\}
• There are \( \binom{n+1}{k} \) subsets of T with k elements: some include a, the rest don’t.
• There are \( \binom{n}{k} \) ways to choose k elements from S (subsets of T without a)
• There are \( \binom{n}{k-1} \) ways to choose k-1 elements from S
  – Add a to each of these, corresponding to subsets of T with a.
• Therefore, \( \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \)
Another Combination Property

• Prove: \( \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 \)

\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k}
\]

• Suppose we have a class of \(2n\) people, and we need to choose \(n\) people to get an A.

• We can divide the room up into the men (there are \(n\) of them) and the women (also \(n\)).

• The \(k\)th term in the summation is the number of ways to assign A’s, so that exactly \(k\) men get an A and \(n-k\) women get an A.

• If we add up over all \(k\), this will give us the total number of ways to assign A’s.
DECOMPOSING TO AND COMBINING ANSWERS TO SUBPROBLEMS
Addition Rule Motivation

• In a version of BASIC, variables can be one or two alphanumeric characters (lower-case and upper-case is not distinguished). The first character must be a letter. How many different variable names are there?
Addition Rule

• In a version of BASIC, variables can be one or two alphanumeric characters (lower-case and upper-case is not distinguished). The first character must be a letter. How many different variable names are there?
  \[26 + 26 \times 36 = 962\]

• The Addition Rule states that if a task can be done in either one of \(n_1\) ways or one of \(n_2\) ways, and there is NO overlap between these groups, then the number of ways is \(n_1 + n_2\), or \(|A \cup B| = |A| + |B|\)

Diagram:
- 1 letter identifier
- 2 letter identifiers
- \(A\)
- \(B\)
Subtraction Rule Motivation

• How many bit strings of length 8 either start with a 1 or end with two 0s?

• The company Grinding Gear Games has 350 job applicants. 220 are CSCI majors, 147 are BUAD majors, and 51 are double majors in both. How many applicants are neither?
Subtraction Rule

• How many bit strings of length 8 either start with a 1 or end with two 0s?
  ➢ $2^7 + 2^6 - 2^5 = 160$

• The company Grinding Gear Games has 350 job applicants. 220 are CSCI majors, 147 are BUAD majors, and 51 are double majors in both. How many applicants are neither?
  ➢ $350 - (220 + 147 - 51) = 34$

• The **Subtraction Rule** states that if a task can be done in either one of $n_1$ ways or one of $n_2$ ways, and there is an overlap between these two methods of $n_3$ common ways, then the number of ways to do the task is $n_1 + n_2 - n_3$, or $|A \cup B| = |A| + |B| - |A \cap B|$
The Subtraction Rule

• Also, known as the Principle of Inclusion / Exclusion (aka PIE)

Overlap will be counted twice

How would you count the ways to solve a task consisting of 3 potentially overlapping subtasks?
The Subtraction Rule

• Also, known as the Principle of Inclusion / Exclusion (aka PIE)

How would you count the ways to solve a task consisting of 3 potentially overlapping subtasks?

\[ |A| + |B| + |C| - |AB| - |AC| - |BC| + |ABC| \]

Overlap will be counted twice
Fundamental Rule Practice (1)

• How many 7-digit phone numbers are there, if they cannot start with a 0, a 1, or the sequence 911?

• How many odd 4-digit numbers are there with no leading zeroes, and no repeated digits?
Fundamental Rule Practice (1)

• How many 7-digit phone numbers are there, if they cannot start with a 0, a 1, or the sequence 911?
  ➢ $8 \cdot 10^6 - 10^4 = 7990000$

• How many odd 4-digit numbers are there with no leading zeroes, and no repeated digits?
  ➢ $5 \cdot 8 \cdot 8 \cdot 7 = 2240$
Fundamental Rule Practice (2)

• A computer system requires a password between 6 and 8 alphanumeric characters (lower-case and upper-case is not distinguished). At least one character must be a digit. How many different passwords are there?
Fundamental Rule Practice (2)

- A computer system requires a password between 6 and 8 alphanumeric characters (lower-case and upper-case is not distinguished). At least one character must be a digit. How many different passwords are there?
  - $36^6 - 26^6 + 36^7 - 26^7 + 36^8 - 26^8$
  - Why is it not $10*36^5 + 10*36^6 + 10*36^7$?
  - Why is it not $6*10*36^5 + 7*10*36^6 + 8*10*36^7$?
PERMUTATIONS AND COMBINATIONS WITH OR WITHOUT REPLACEMENT
Motivation for Permutations with Repetition

• A message on a Twitter-like service must consist of exactly \( r \) characters. Because it is the internet, people only communicate using capital letters, spaces, and the exclamation mark.

• How many different messages can one post on this service?
Motivation for Permutations with Repetition

• A message on a Twitter-like service must consist of exactly \( r \) characters. Because it is the internet, people only communicate using capital letters, spaces, and the exclamation mark.

• How many different messages can one post on this service?
  – \( 28^r \)

• The number of \( r \)-permutations of a set of \( n \) objects, where repetition is allowed, is \( n^r \).
Motivation for Combinations with Repetition

• How many ways are there to select 5 bills from a cash box containing 1, 2, 5, 10, 20, 50, and 100 dollar bills (7 denominations)?
  – Is it just $7^5$?
Combinations with Repetition  
(Stars and Bars)

• How many ways are there to select 5 bills from a cash box containing 1, 2, 5, 10, 20, 50, and 100 dollar bills (7 denominations)?

*       |                |       *      |                  |         *          |       **      |
1                  2                5              10                 20                 50              100

• We need to write 7-1=6 bars (cash box dividers) and 5 stars (the bills) in some order (choose the locations for the *s or bars)

\[
\binom{11}{5} = \binom{11}{6} = \begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ & \_ \\
* & | & | & * & | & | & * & | & * & | \\
| & | & * & * & * & | & | & * & | & * \\
| & | & | & | & | & | & | & | & | \\
\end{array}
\]

• The number of r-combinations of a set of n objects, where repetition is allowed, is \( \binom{n-1+r}{r} = \binom{n-1+r}{n-1} \)
Summary of Counting (Subsets) Taxonomy

- Summary based on replacement and ordering

<table>
<thead>
<tr>
<th></th>
<th>No Replacement</th>
<th>Replacement</th>
</tr>
</thead>
</table>
| **Order matters** | $nP_k = \frac{n!}{(n-k)!}$  
  n total options, permute k  
  Special case: Permute all n:  
  $nP_n = \frac{n!}{(n-n)!} = n!$ | For replacement:  
  $k^n$  
  Length n, k options  
  (Base-n combinations) |
| **Order does NOT matter** | \( \binom{n}{k} = \frac{n!}{k! (n-k)!} \)  
  n total options, choose k | \( \binom{n + k - 1}{k} = \binom{n + k - 1}{n - 1} \)  
  Stars and Bars |
MULTISETS, ELEMENTS, BOXES, AND DISTINGUISHABILITY
Motivation for Multisets

• A multiset is a set which may contain duplicates
  – The number of occurrences of each element is its \textit{multiplicity}
  – E.g. The multiplicity of 'S' in "SUCCESS" is 3

• How many different strings can be made by reordering the letters of SUCCESS?
Multisets (Approach 1)

• How many different strings can be made by reordering the letters of SUCCESS?
  ➢ What would the answer be if all the characters were unique ("1234567")?
  ➢ But now realize 1 6 7 are interchangeable since they represent the same letter S and 3 and 4 are interchangeable since they represent the letter C

• The number of permutations of n objects, where \( n_i \) is the multiplicity (are indistinguishable objects) of type i, is

\[
\frac{n!}{\prod_{i=1}^{k} n_i!}
\]

\[\begin{array}{cccccccc}
S & U & C & C & E & S & S \\
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}\]
Multisets (Approach 2)

• Take each "type" and choose which locations will have that type
• For SUCCESS, choose the 3 places for S, 2 places for C, etc.?

\[
\binom{7}{3} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1} = \frac{7!}{3! \cdot 4!} \cdot \frac{4!}{2! \cdot 2!} \cdot \frac{2!}{1! \cdot 1!} \cdot \frac{1!}{0! \cdot 1!}
\]

\[
= \frac{7!}{3! \cdot 2! \cdot 1! \cdot 1!}
\]

• The number of permutations of \( n \) objects, where \( n_i \) are indistinguishable objects of type \( i \), is

\[
\frac{n!}{\prod_{i=1}^{k} n_i!}
\]
Distinguishable Objects over Distinguishable Boxes

• 4 people are playing Poker, wherein each player receives 5 cards from a standard deck of 52 distinct cards. How many ways are there to deal out hands?

\[
\binom{52}{5} \cdot \binom{47}{5} \cdot \binom{42}{5} \cdot \binom{37}{5}
\]

\[
= \frac{52!}{5! \cdot 47!} \cdot \frac{47!}{5! \cdot 42!} \cdot \frac{42!}{5! \cdot 37!} \cdot \frac{37!}{5! \cdot 32!}
\]

\[
= \frac{52!}{5! \cdot 5! \cdot 5! \cdot 5! \cdot 32!}
\]

• The number of ways to distribute \( n \) distinguishable objects into \( k \) distinguishable boxes so that \( n_i \) objects are placed in box \( i \) is:

\[
\frac{n!}{\prod_{i=1}^{k} n_i!}
\]

Transform the problem to permutations with repetition

<table>
<thead>
<tr>
<th>2♣</th>
<th>2♣</th>
<th>2♥</th>
<th>2♦</th>
<th>3♣</th>
<th>3♣</th>
<th>3♥</th>
<th>3♦</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>D</td>
<td>P4</td>
<td>D</td>
<td>P2</td>
<td>P1</td>
<td>D</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

Bijection between cards and player hands (or D=Deck). How many permutations of the 5 P1s, 5 P2s, ..., 32 D (Deck) exist?
Indistinguishable Objects over Distinguishable Boxes

• There are 8 poker players, and there are 10 one-hundred-dollar chips distributed amongst them. How many ways could the chips be distributed?

• The number of ways to distribute \( r \) indistinguishable objects into \( n \) distinguishable boxes is
Indistinguishable Objects over Distinguishable Boxes

- There are 8 poker players, and there are 10 one-hundred-dollar chips distributed amongst them. How many ways could the chips be distributed?
  
  $$\binom{17}{10}$$

- The number of ways to distribute \( r \) indistinguishable objects into \( n \) distinguishable boxes is
  
  $$\binom{n-1+r}{r}$$
Indistinguishable Objects over Nonempty Distinguishable Boxes

- If we want to place $r$ indistinguishable objects into $n$ distinguishable boxes, but every box must have at least one element, we can start by placing one object in every box.

- Now we need to place $r-n$ indistinguishable objects into $n$ distinguishable boxes.
  - This is the same problem as the previous one, now!
  - $\binom{r-n+(n-1)}{r-n} = \binom{r-1}{r-n}$
Practice

• How many ways can ANAGRAM be rearranged?

• How many ways are there to select 4 pieces of fruit from a bowl containing apples, oranges, and pears?

• How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where $x_1$, $x_2$, and $x_3$ are nonnegative integers?
Practice

• How many ways can ANAGRAM be rearranged?
  \[- \frac{7!}{3!}\]

• How many ways are there to select 4 pieces of fruit from a bowl containing apples, oranges, and pears?
  – Combinations, with repetition
  \[- \binom{6}{4}\]

• How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where $x_1$, $x_2$, and $x_3$ are nonnegative integers?
  – Indistinguishable objects into distinguishable boxes
  \[- \binom{13}{11}\]
Distinguishable Objects over Indistinguishable Boxes

• 4 different roommates are playing a multi-player game, and they’re distributed over 3 indistinguishable servers. How many ways are there to distribute them?
Distinguishable Objects over Indistinguishable Boxes

• 4 different roommates are playing a multi-player game, and they’re distributed over 3 indistinguishable servers. How many ways are there to distribute them?
  – Consider breaking this problem into non-overlapping subproblems and use the addition rule. For each case, consider the combinations that can be made.
    – (4,0,0) = 1
    – (3,1,0) = 4
    – (2,1,1) = 6
    – (2,2,0) = 3
    – Total = 14
  – There is no closed-form formula for this problem.
Indistinguishable Objects over Indistinguishable Boxes

• The company just worries about server-load, not who plays where.

• How many ways can you distribute 6 faceless players over 4 indistinguishable servers?
  – Again, break into case but now each case does not need to consider the different combinations that can be made since the objects are indistinguishable
    – (6,0,0,0)
    – (5,1,0,0) (4,2,0,0) (3,3,0,0)
    – (4,1,1,0) (3,2,1,0) (2,2,2,0)
    – (3,1,1,1) (2,2,1,1)

• Total = 9

• There is no closed-form formula for this problem.
# Summary Of Distributing Elements

- Distributing $n$ items over $k$ boxes

<table>
<thead>
<tr>
<th></th>
<th>Distinguishable Boxes</th>
<th>Indistinguishable Boxes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ Indistinguishable Elements</td>
<td>Stars and Bars [ \binom{n + k - 1}{k - 1} ]</td>
<td>Break into cases: [ x_1 + x_2 + \cdots + x_k = n ] (i.e. way to sum $k$ non-neg integers to equal $n$) remembering order of $x_i$ doesn't matter</td>
</tr>
<tr>
<td></td>
<td>If at least 1 per box: [ \binom{n - 1}{k - 1} ]</td>
<td>No closed-form formula</td>
</tr>
<tr>
<td>$n$ Distinguishable Elements</td>
<td>[ \frac{n!}{\prod_{i} n_i!} ]</td>
<td>Break into cases: [ x_1 + x_2 + \cdots + x_k = n ] For each case, determine number of orderings (ways to choose $x_i$) but ordering of $x_1 \ldots x_k$ doesn't matter</td>
</tr>
<tr>
<td></td>
<td>$n_i$ elements in a boxes for $i = 1 \ldots k$</td>
<td>No closed-form formula</td>
</tr>
</tbody>
</table>