CSCI 104
Recursion –
Combinations & Backtracking
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Recursion in CS 104

• Problem in which the solution can be expressed in terms of itself (usually a smaller instance/input of the same problem) and a base/terminating case

• Recursion is a key concept in this course
  – But it rarely comes easily to students. You must work at it!

• Many problems that would be VERY difficult to solve without recursion (i.e. only loops) have extremely elegant solutions to problems
  – Learn to look for those elegant solutions
  – In this class, assume the recursive approach has an elegant/simple solution
  – If you find yourself writing a large, complex recursive solution, assume you are doing something you should not!
    • Stop and reconsider how it should be done
Simple vs. Multiple Recursion

• "Simple" recursion refers to functions that contain just **ONE recursive call**
  – Can be head or tail recursion (explained soon)
  – Can easily be replaced by a loop

• The power of recursion usually comes when the function makes **2 OR MORE recursive calls** (aka "multiple recursion")
  – Elegant recursive solutions that would be **MUCH harder to implement iteratively**
    (usually need a separate stack data structure)

• We'll focus on **multiple recursion**

```cpp
void print(Item* p) {
    if(p == NULL) return;
    else {
        cout << p->val << endl;
        print(p->next);
    }
}
```

**Simple Recursion**
(1 recursive call)

```cpp
void postorder(TNode* t) {
    if(t == NULL) return
    postorder(t->left)
    postorder(t->right)
    process(t) // print val.
}
```

**Multiple Recursion**
(2 or more recursive calls)
Steps to Formulating Recursive Solutions

1. Solve a few instances of the problem to discover the recursive structure

2. Identify how the problem can be decomposed into smaller problems of the same form
   – Does solving the problem on an input of smaller value or size help formulate the solution to the larger

3. Identify the base case
   – An input for which the answer is trivial

4. Assume the recursive call for the smaller problem "magically" computes the correct solution(s) to those problem(s) and identify how to combine those solution(s) from the smaller problem(s) into the solution for the larger problem
Towers of Hanoi Problem

- Problem Statements: Move n discs from source pole to destination pole (with help of a 3rd alternate pole)
  - Cannot place a larger disc on top of a smaller disc
  - Can only move one disc at a time
Observation 1

- Observation 1: Disc 1 (smallest) can always be moved
- Solve the n=2 case:

\[
\begin{array}{ccc}
A \text{ (src)} & B \text{ (dst)} & C \text{ (alt)} \\
\begin{array}{c}
1 \\
2
\end{array} & \begin{array}{c}
\text{ } \\
\text{ }
\end{array} & \begin{array}{c}
\text{ } \\
\text{ }
\end{array}
\end{array}
\]

Start

Move 1 from src to alt

Move 2 from src to dst

Move 1 from alt to dst
Observation 2

• Observation 2: If there is only one disc on the src pole and the dest pole can receive it the problem is trivial.

Move n-1 discs from src to alt

Move disc n from src to dst

Move n-1 discs from alt to dst
Recursive solution

• But to move n-1 discs from src to alt is really a smaller version of the same problem with
  – n => n-1
  – src=>src
  – alt =>dst
  – dst=>alt

• Towers(n,src,dst,alt)
  – Base Case: n==1  // Observation 1: Disc 1 always movable
    • Move disc 1 from src to dst
  – Recursive Case:  // Observation 2: Move of n-1 discs to alt & back
    • Towers(n-1,src,alt,dst)
    • Move disc n from src to dst
    • Towers(n-1,alt,dst,src)
Recursive Box Diagram

Towers Function Prototype

Towers(disc,src,dst,alt)

Towers(3,a,b,c)
  - Move D=3 a to b
  - Towers(2,c,b,a)
    - Move D=2 c to b
    - Towers(1,a,b,c)
      - Move D=1 a to b
      - Towers(1,b,c,a)
        - Move D=1 b to c
        - Towers(1,c,a,b)
          - Move D=1 c to a
          - Towers(1,a,b,c)
            - Move D=1 a to b
            - Towers(1,a,b,c)
              - Move D=1 a to b
GENERATING ALL COMBINATIONS
Recursion's Power

• The power of recursion often comes when each function instance makes *multiple* recursive calls

• As you will see this often leads to an exponential number of "combinations" being generated/explored in an easy fashion
Binary Combinations

• If you are given the value, n, and a string with n characters could you generate all the combinations of n-bit binary?

• Do so recursively!

Exercise: bin_combo_str
Recursion and DFS

- Recursion forms a kind of Depth-First Search

```cpp
// user interface
void binCombos(int len) {
    binCombos("", len);
}

// helper-function
void binCombos(string prefix, int len) {
    if(prefix.length() == len )
        cout << prefix << endl;
    else {
        // recurse
        binCombos(prefix + "0", len);
        binCombos(prefix + "1", len);
    }
}
```
Generating All Combinations

- Recursion offers a simple way to generate all $N$-length combinations of from a set of options, $S$
  
  - Example: Generate all 2-digit decimal numbers ($N=2$, $S=\{0,1,\ldots,9\}$)

```cpp
void NDigDecCombos(string data, int n)
{
    if(data.size() == n )
        cout << data;
    else {
        for(int i=0; i < 10; i++){
            // recurse
            NDigDecCombos(data+(char)('0'+i),n);
        }
    }
}
```

Options

- $N = \text{length}$
Another Exercise

• Generate all string combinations of length n from a given list (vector) of characters

```cpp
#include <iostream>
#include <string>
#include <vector>
using namespace std;

void all_combos(vector<char>& letters, int n) {
    // ???
}

int main() {
    vector<char> letters = {'U', 'S', 'C'};
    all_combos(letters, 4);
    return 0;
}
```

Options

| U | S | C |

Use recursion to walk down the 'places'
At each 'place' iterate through & try all options
Recursion and Combinations

• Recursion provides an elegant way of generating all **n**-length combinations of a set of values, **S**.
  – Ex. Generate all length-**n** combinations of the letters in the set **S**={'U','S','C'} (i.e. for **n**=2: UU, US, UC, SU, SS, SC, CU, CS, CC)

• General approach:
  – Need some kind of **array/vector/string** to store partial answer as it is being built
  – Each recursive call is only responsible for one of the **n** "places" (say location, **i**)  
  – The function will iteratively (loop) try each option in **S** by setting location **i** to the current option, then recurse to handle all remaining locations (**i**+1 to **n**)
    • Remember you are responsible for only one location
  – Upon return, try another option value and recurse again
  – Base case can stop when all **n** locations are set (i.e. recurse off the end)
  – Recursive case returns after trying all options
Exercises

- bin_combos_str
- Zero_sum
- Prime_products_print
- Prime_products
- basen_combos
- all_letter_combos
Recursive Backtracking Search

• Recursion allows us to "easily" enumerate all solutions/combinations to some problem
• Backtracking algorithms are often used to solve constraint satisfaction problems or optimization problems
  – Find (the best) solutions/combinations that meet some constraints
• Key property of backtracking search:
  – Stop searching down a path at the first indication that constraints won't lead to a solution
• Many common and important problems can be solved with backtracking approaches
• Knapsack problem
  – You have a set of products with a given weight and value. Suppose you have a knapsack (suitcase) that can hold N pounds, which subset of objects can you pack that maximizes the value.
  – Example:
    • Knapsack can hold 35 pounds
    • Product A: 7 pounds, $12 ea.  Product B: 10 pounds, $18 ea.
    • Product C: 4 pounds, $7 ea.  Product D: 2.4 pounds, $4 ea.
• Other examples:
  – Map Coloring, Satisfiability, Sudoku, N-Queens
N-Queens Problem

• Problem: How to place N queens on an NxN chess board such that no queens may attack each other

• Fact: Queens can attack at any distance vertically, horizontally, or diagonally

• Observation: Different queen in each row and each column

• Backtrack search approach:
  – Place 1st queen in a viable option then, then try to place 2nd queen, etc.
  – If we reach a point where no queen can be placed in row i or we've exhausted all options in row i, then we return and change row i-1
8x8 Example of N-Queens

• Now place 2\textsuperscript{nd} queen
8x8 Example of N-Queens

• Now place others as viable
• After this configuration here, there are no locations in row 6 that are not under attack from the previous 5
• BACKTRACK!!!
8x8 Example of N-Queens

• Now place others as viable
• After this configuration here, there are no locations in row 6 that is not under attack from the previous 5
• So go back to row 5 and switch assignment to next viable option and progress back to row 6
8x8 Example of N-Queens

• Now place others as viable
• After this configuration here, there are no locations in row 6 that is not under attack from the previous 5
• Now go back to row 5 and switch assignment to next viable option and progress back to row 6
• But still no location available so return back to row 5
8x8 Example of N-Queens

• Now place others as viable
• After this configuration here, there are no locations in row 6 that is not under attack from the previous 5
• Now go back to row 5 and switch assignment to next viable option and progress back to row 6
• But still no location available so return back to row 5
• But now no more options for row 5 so return back to row 4
• BACKTRACK!!!!
8x8 Example of N-Queens

- Now place others as viable
- After this configuration here, there are no locations in row 6 that is not under attack from the previous 5
- Now go back to row 5 and switch assignment to next viable option and progress back to row 6
- But still no location available so return back to row 5
- But now no more options for row 5 so return back to row 4
- Move to another place in row 4 and restart row 5 exploration
8x8 Example of N-Queens

- Now place others as viable
- After this configuration here, there are no locations in row 6 that is not under attack from the previous 5
- Now go back to row 5 and switch assignment to next viable option and progress back to row 6
- But still no location available so return back to row 5
- But now no more options for row 5 so return back to row 4
- Move to another place in row 4 and restart row 5 exploration
8x8 Example of N-Queens

• Now a viable option exists for row 6
• Keep going until you successfully place row 8 in which case you can return your solution
• What if no solution exists?
8x8 Example of N-Queens

• Now a viable option exists for row 6
• Keep going until you successfully place row 8 in which case you can return your solution
• What if no solution exists?
  – Row 1 queen would have exhausted all her options and still not find a solution
Backtracking Search

- Recursion can be used to generate all options
  - 'brute force' / test all options approach
  - Test for constraint satisfaction only at the bottom of the 'tree'

- But backtrack search attempts to 'prune' the search space
  - Rule out options at the partial assignment level

Brute force enumeration might test only when a complete assignment is made (i.e. all 4 queens on the board)
N-Queens Solution Development

• Let's develop the code
• 1 queen per row
  – Use an array where index represents the queen (and the row) and value is the column
• Start at row 0 and initiate the search [i.e. search(0) ]
• Base case:
  – Rows range from 0 to n-1 so STOP when row == n
  – Means we found a solution
• Recursive case
  – Recursively try all column options for that queen
  – But haven't implemented check of viable configuration...

```
int *q;  // pointer to array storing each queens location
int n;   // number of board / size

void search(int row)
{
  if(row == n)
    printSolution(); // solved!
  else {
    // remember q[row] is the column
    for(q[row]=0; q[row]<n; q[row]++){
      search(row+1);
    }
    // alternatively
    // for(int col = 0; col < n; col++){
    //   q[row] = col;
    //   search(row+1);
    // }
  }
```

Index = Queen i in row i
q[i] = column of queen i
N-Queens Solution Development

• To check whether it is safe to place a queen in a particular column, let's keep a "threat" 2-D array indicating the threat level at each square on the board
  – Threat level of 0 means SAFE
  – When we place a queen we'll update squares that are now under threat
  – Let's name the array 't'

• Dynamically allocating 2D arrays in C/C++ doesn't really work
  – Instead conceive of 2D array as an "array of arrays" which boils down to a pointer to a pointer

```
int *q;  // pointer to array storing each queens location
int n;   // number of board / size
int **t; // thread 2D array

main()
{
  q = new int[n];
  t = new int*[n];
  for(int i=0; i < n; i++){
    t[i] = new int[n];
    for(int j = 0; j < n; j++){
      t[i][j] = 0;
    }
  }
  search(0); // start search
  // deallocate arrays
  return 0;
}
```

Each entry is int *

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

0 1 2 3

Index = Queen i in row i
q[i] = column of queen i

```
0 1 2 3
0 1 1 1
1 1 0 0
2 0 1 0
3 0 0 1
```

Allocated on line 08

<table>
<thead>
<tr>
<th>00</th>
<th>01</th>
<th>02</th>
<th>03</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>2c0</td>
<td>1b4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Each allocated on an iteration of line 10

<table>
<thead>
<tr>
<th>00</th>
<th>01</th>
<th>02</th>
<th>03</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>2c0</td>
<td>1b4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

t[2] = 0x1b4

1b4

Thus t is int **

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N-Queens Solution Development

• After we place a queen in a location, let's check that it has no threats
• If it's safe then we update the threats (+1) due to this new queen placement
• Now recurse to next row
• If we return, it means the problem was either solved or more often, that no solution existed given our placement so we remove the threats (-1)
• Then we iterate to try the next location for this queen

```c
int *q; // pointer to array storing // each queens location
int n; // number of board / size
int **t; // n x n threat array
void search(int row)
{
    if(row == n)
        printSolution(); // solved!
    else {
        for(q[row]=0; q[row]<n; q[row]++){
            // check that col: q[row] is safe
            if(t[row][q[row]] == 0){
                // if safe place and continue
                addToThreats(row, q[row], 1);
                search(row+1); // if return, remove placement
                addToThreats(row, q[row], -1);
            }
        }
    }
}
```

<table>
<thead>
<tr>
<th>q[i] = column of queen i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index = Queen i in row i</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
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<td>2</td>
<td>0</td>
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</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Safe to place queen in upper left</th>
<th>Now add threats</th>
<th>Upon return, remove threat and iterate to next option</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 1 2 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1 1 1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1 1 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 0 1 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 0 0 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 0 0 0</td>
<td></td>
</tr>
</tbody>
</table>
addToThreats Code

• Observations
  – Already a queen in every higher row so addToThreats only needs to deal with positions lower on the board
    • Iterate row+1 to n-1
  – Enumerate all locations further down in the same column, left diagonal and right diagonal
  – Can use same code to add or remove a threat by passing in change
• Can't just use 2D array of booleans as a square might be under threat from two places and if we remove 1 piece we want to make sure we still maintain the threat

```
void addToThreats(int row, int col, int change)
{
    for(int j = row+1; j < n; j++){
        // go down column
        t[j][col] += change;
        // go down right diagonal
        if( col+(j-row) < n )
            t[j][col+(j-row)] += change;
        // go down left diagonal
        if( col-(j-row) >= 0)
            t[j][col-(j-row)] += change;
    }
}
```
N-Queens Solution

```c++
int *q; // queen location array
int n; // number of board / size
int **t; // n x n threat array

int main()
{
    q = new int[n];
t = new int*[n];
    for(int i=0; i < n; i++){
        t[i] = new int[n];
        for(int j = 0; j < n; j++){
            t[i][j] = 0;
        }
    }
    // do search
    if( !search(0) )
        cout << "No sol!" << endl;
    // deallocate arrays
    return 0;
}

void addToThreats(int row, int col, int change)
{
    for(int j = row+1; j < n; j++){
        // go down column
        t[j][col] += change;
        // go down right diagonal
        if( col+(j-row) < n )
            t[j][col+(j-row)] += change;
        // go down left diagonal
        if( col-(j-row) >= 0 )
            t[j][col-(j-row)] += change;
    }
}

bool search(int row)
{
    if(row == n){
        printSolution(); // solved!
        return true;
    }
    else {
        for(q[row]=0; q[row]<n; q[row]++){
            // check that col: q[row] is safe
            if(t[row][q[row]] == 0){
                // if safe place and continue
                addToThreats(row, q[row], 1);
                bool status = search(row+1);
                if(status) return true;
                // if return, remove placement
                addToThreats(row, q[row], -1);
            }
            return false;
        }
    }
    return false;
}
```

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General Backtrack Search Approach

- Select an item and set it to one of its options such that it meets current constraints
- Recursively try to set next item
- If you reach a point where all items are assigned and meet constraints, done...return through recursion stack with solution
- If no viable value for an item exists, backtrack to previous item and repeat from the top
- If viable options for the 1st item are exhausted, no solution exists
- Phrase:
  - Assign, recurse, unassign

---

**General Outline of Backtracking Sudoku Solver**

```c
bool sudoku(int **grid, int r, int c) {
    if( allSquaresComplete(grid) )
      return true;
}
// iterate through all options
for(int i=1; i <= 9; i++){
    grid[r][c] = i;
    if( isValid(grid) ){
      bool status = sudoku(...);
      if(status) return true;
    }
}
return false;
```

Assume r,c is current square to set and grid is the 2D array of values
Runtime of All Combinations

- \( T(n_r, n_c) = \) ________________
- \( T(0, n_c) = 1 \)

```c
int *q; // pointer to array storing
         // each queens location
int n;   // number of board / size

void search(int row)
{
    if (row == n)
        printSolution(); // solved!
    else {
        // remember q[row] is the column
        for (q[row] = 0; q[row] < n; q[row]++){
            search(row + 1);
        }
    }
}
```
SOLUTIONS
Recursion and DFS

• Recursion forms a kind of Depth-First Search

```cpp
// user interface
void binCombos(int len)
{
    binCombos("", len);
}

// helper-function
void binCombos(string prefix, int len)
{
    if(prefix.length() == len )
        cout << prefix << endl;
    else {
        // recurse
        binCombos(prefix+"0" len);
        // recurse
        binCombos(prefix+"1", len);
    }
}
```

Generally: Recursion must perform the same code sequence for each item. Where we need variation, use 'if' statements.