Discrete Ricci Flow for Detecting Gaps in Adiabatic Quantum Processes

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Overview

Outline

Adiabatic Quantum Processes (AQP)

- In an adiabatic quantum process, the system evolves from the ground state of an initial Hamiltonian H_0 to the ground state of a final Hamiltonian H_1
- $$H(s) = (1-s)\,H_0 + s\,H_1, \quad s \in [0,1],$$
- $E_0(s) \leq E_1(s) \leq E_2(s) \leq \ldots$: energy levels of $H(s)$
- $E_0(s=1)$ of H_1 encodes the solution of a hard optimization task
- The **spectral gap** between the ground and first excited state is

$$\Delta_{\min} = \min_{0 \leq s \leq 1} \Delta_{1,0}(s), \quad \Delta_{1,0}(s) = E_1(s) - E_0(s)$$

- The runtime of the AQP is inversely proportional to the spectral gap

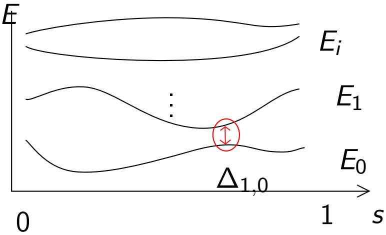
Spectral Gap

- Location and size of the spectral gap $\Delta_{\min}$ are usually unknown.

Challenge

Detecting where this gap occurs is critical to improve performance and runtime of the AQP.

- Gaps often occur at **avoided crossings** in the energy levels.
- Hard to locate without full diagonalization of $H(s)$.
- We therefore look for an indirect, **geometric** way to detect gaps.



Solution

Use curvature transport and Ricci flow on the eigenvalue curves $E_i(s)$ to detect where the gap becomes small.

Outline

Curvature & Ricci Flow

Curvature

- Curvature measures how a geometric space deviates from being flat
- In Riemannian geometry, quantifies how geodesic paths converge (positive curvature) or diverge (negative curvature)
- Measures local geometry of spaces

Ricci Flow:

- A geometric evolution equation for smoothing irregularities
- $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}(g)$

Curvature Transport Method

Landscape of $E_i(s)$:

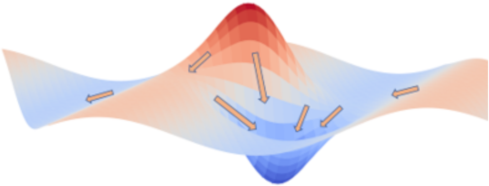
- Curvature of $E_i(s)$
- Curvature $\equiv$ "mass"
- Max. "mass" transport rate at max. curvature of $E_i(s)$

Earth moving metaphor:

- Sand transport from hills to valleys

Heat Diffusion metaphor:

- Max temperature $\Rightarrow$ largest heat flux



Solution

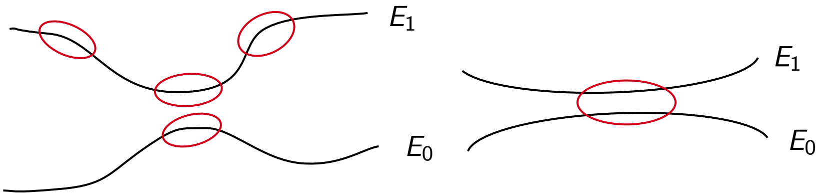
Take advantage of sharpness of "valleys" and "ridges" and detect high curvature by curvature transport

Outline

Gap Detection

Spectral Gap:

- High curvature regions in $E_i(s)$ likely indicate a gap
- High curvature regions correspond to areas of high curvature transport
- Ricci flow serves as "magnifying lens"



Background

Modify scheduling:

- $H(s) = p_0(s)H_0 + p_1(s)H_1$
- Global properties of scheduling:

$$p_0(s) = \cos\frac{\pi s}{2}, p_1(s) = \sin\frac{\pi s}{2}, \quad s \in [0, 4]$$

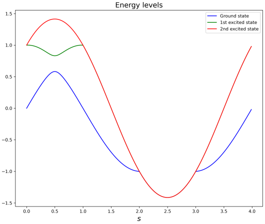
- $H(\vartheta) = \cos(\vartheta)H_0 + \sin(\vartheta)H_1, \quad \vartheta \triangleq \frac{\pi s}{2}, \quad \vartheta \in [0, 2\pi]$
  - Extend path from linear to cyclic
  - Path is smooth
- Energy levels E_k are eigenvalues: $\lambda_k(H(\vartheta)) \triangleq \lambda_k(\vartheta)$

4-vertex Theorem (Tabachnikov)

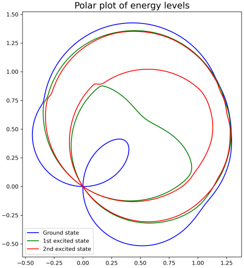
Any smooth, simple, closed curve in the plane has at least 4 vertices.

Polar Plot:

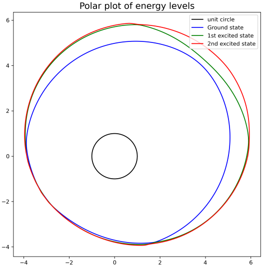
- Represents energy levels as closed curves.
- Curve defined by: $x(\vartheta) \mapsto \lambda(\vartheta)\cos(\vartheta)$ and $y(\vartheta) \mapsto \lambda(\vartheta)\sin(\vartheta)$



"Classical plot"



"Genuine plot"



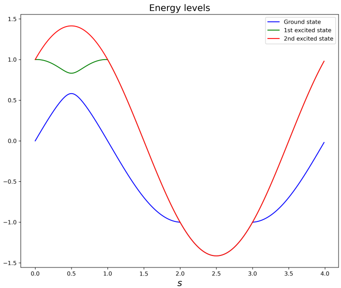
"Offset plot"

- "Offset" plot is "subjective" representation

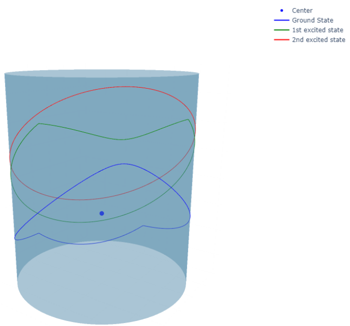
Cylinder plot

Cylinder Plot:

- Energy levels mapped onto a cylindrical surface
- $x(\vartheta) \mapsto \cos(\vartheta)$, $y(\vartheta) \mapsto \sin(\vartheta)$, and $z(\vartheta) \mapsto \lambda(\vartheta)$



"Genuine plot"



"Cylinder plot"

- Avoids "offseting" $\Rightarrow$ "objective" representation

Discrete Ricci Flow on Energy Level Curves

- Discretize curve into vertices v_i
- At each vertex v_i:
  - Curvature K_i
  - Conformal factors u_i which modify the metric
- Ricci/Yamabe flow:

$$\frac{du_i}{dt} = -K_i(t)u_i(t), \quad u_i(0) = 1$$

$$\frac{dK_{u(t)}}{dt} = \mathcal{L}\left(K_{u(t)}\right), \quad K_{u(0)}(v_i) = K(v_i)$$

- $\mathcal{L}(.)$ is the graph Laplacian

Gap detection

Vertices with high $\frac{dK}{dt} \Rightarrow$ high curvature transport $\Rightarrow$ spectral gap regions

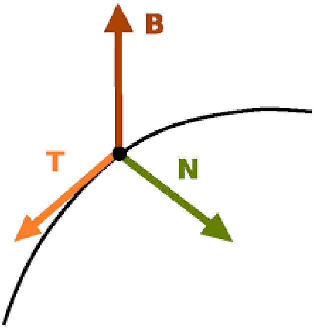
Continuous Ricci Flow

Polar Plot Curve

- Curve in $\mathcal{R}^2$
- Curvature: $\kappa = \frac{d\vec{T(s)}}{ds}$
- $\kappa(\vartheta) = \frac{-\lambda''(\vartheta)\lambda(\vartheta)+2\lambda'(\vartheta)^2+\lambda(\vartheta)^2}{(\lambda'(\vartheta)^2+\lambda(\vartheta)^2)^{3/2}}$

Cylinder Plots:

- Curve in $\mathcal{R}^3$
- $\frac{dT}{ds} = \kappa N, \quad \frac{dN}{ds} = -\kappa T + \tau B, \quad \frac{dB}{ds} = -\tau N$
- $\kappa(\vartheta) = \frac{r\sqrt{\lambda''(\vartheta)^2+\lambda'(\vartheta)^2+r^2}}{(\lambda'(\vartheta)^2+r^2)^{1.5}}$
- $\tau(\vartheta) = \frac{\lambda'''(\vartheta)+\lambda'(\vartheta)}{\lambda''(\vartheta)^2+\lambda'(\vartheta)^2+r^2}$



Ricci Flow:

$$u(\vartheta, t)u_t(\vartheta, t) = \begin{cases} 1/r - \kappa(\vartheta, t), \\ \tau(\vartheta, t), \end{cases}$$

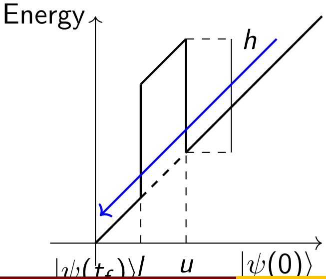
Outline

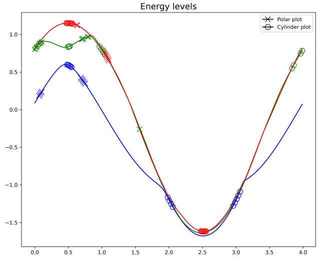
QUBO

- Quadratic Unconstrained Binary Optimization
- Combinatorial optimization problem on the graph $G(V_f, \mathcal{E}_f)$

Hamiltonians:

- $H_0 = \sum_i^n S_x^{(i)}$
- $H_1 = H_w + H_b + H_d$
  - $H_w = \sum_i^n S_z^{(i)}$
  - $H_b = \frac{h}{2}(\text{sign}\{H_w - (l - \frac{1}{2})I\} - \text{sign}\{H_w - (u + \frac{1}{2})I\})$
  - $H_d = \epsilon_d \sum_i^n r_i S_y^{(i)}$





Results: QUBO - Continuous Flow

QUBO

- Combinatorial optimization problem on the graph $G(V_f, \mathcal{E}_f)$

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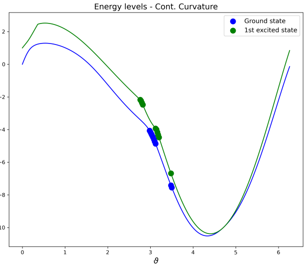

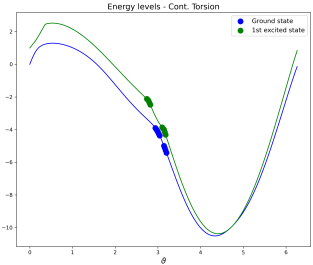

Outline

Summary

- Introduced a novel geometric method to detect spectral gaps in AQPs
- Used Ricci flow to track high curvature events on energy-level curves
- Introduced the polar and cylinder plot to apply the Ricci flow
- Demonstrated results on a QUBO with barrier AQP:
  - Discrepancies between polar and cylinder discrete plots
  - Continuous flow more accurate than discrete
- Cylinder embedding provides most stable results.

Thank you!