Interference Constrained Network Control Based on Curvature

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Abstract— **This paper proceeds from the premise that the topology of interference constrained wireless networks heavily impacts their node-to-node delay, routing energy, and capacity region. We quantitatively analyze how the discrete Ollivier-Ricci curvature of a network affects the performance metrics of several routing protocols. Since different protocols are optimal relative to different metrics under different topologies, an adaptive control system is proposed that identifies the topology curvature and selects the best protocol under current circumstances subject to user needs. Also, we analyze how sensitive the four routing protocols (Heat Diffusion, Dirichlet, Back Pressure and Shortest Path Routing) under examination are to varying topological environment, as it would commonly be encountered in wireless networks.**

I. Introduction

A Wireless Sensor Network (WSN) typically consists of a large number of low-power computation-capable autonomous nodes. Unlike wired networks where node-to-node delay is usually the only optimization objective, in WSNs power conservation is also a major concern. Sensor nodes usually carry generally irreplaceable power sources, densely deployed within a frequently changing environment topology [1]. This is the reason why a power-aware routing protocol is usually preferred in sensor networks instead of network flooding. However, certain applications still require lower delay, which generally entails a trade-off with power consumption. SEAD (Scalable Energy-Efficient Asynchronous Dissemination [2]) is an example of a protocol that proposes to trade-off between node-to-node delay and energy saving. In addition to protocol dependence, the various conflicting metrics also depend on network topology—more specifically curvature that has become the prime metric in relation to congestion and load balancing [5], stability and capacity region [19], transmission reliability [23], etc.

Our focus here is to develop a numerical, quantitative understanding of the relationship between network topology and delay/routing energy performance. Network topology here is understood in the Ollivier-Ricci curvature sense [19] for the main reason that it regulates the heat flow that our Heat Diffusion and related protocol mimic. Since a WSN is subject to frequent network topology change, we also analyze the impact of topology change on routing performance, and develop a measure of protocol robustness to varying topology. Ultimately, we target a control scheme that dynamically adapts protocols in different network topological environment subject to different user preference.

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II. Different curvatures for different protocols

A. Shortest Path Routing in Wired Networks

It is shown in [3] that, under Shortest Path Routing (SPR) protocol where each node is forwarding packets in accordance with a global routing table, congestion of the network arises as a combination of the LPR protocol and the negative curvature of the network. It is proved that in a Gromov negatively curved space, geodesics with uniformly distributed (source, destination) end points concentrate in, and create congestion at, the "centroid" of the network. The "centroid" is quantitatively defined as a point of maximum betweenness centrality. Asymptotic congestion estimates at the centroid for very large size networks clearly demonstrate the congestion-curvature dependency [4]. As we will show in later sections, network "congestion" in wireless networks is consistent with these wireline results—provided that we use a more proper curvature metric.

B. Heat Diffusion Routing in Wireless Networks

In WSNs, the global topology information is not generally available to every node, so that access to a global routing table is no longer an option. Thus, a dynamic routing protocol referred to as Back-Pressure (BP) [6] routing that acts on local queue backlogs has been proposed. It achieves maximum throughput in the presence of varying network topology without knowing arrival rates nor global topology. BP has been widely investigated, including optimal BP routing in unreliable channels [24], comparing MaxWeight routing with BP [25] and combing BP with LPR [26].

Heat Diffusion (HD) protocol, originally proposed in [7], is also a dynamic routing protocol with the unique feature that it mimics the discrete heat diffusion process with only information from neighboring nodes. It is proved to stabilize the network for any rate matrix in the interior of the capacity region.

The Heat Diffusion protocol, as a variation of BP, is briefly formulated as follows:
At timeslot n, let $Q_i^{(d)}(n)$ denote the number of d-packets (those packets bound to destination $d \in K$) queued at the network layer in node i, and $f_{ij}^{(d)}(n)$ the *actual* number of d-packets transmitted via link ij, constrained by the link capacity $\mu_{ij}(n)$. HD is designed along the same 3-stage process as BP: *weighting-scheduling-forwarding.*

- **HD Weighting:** At each timeslot n and for each link (i, j), the algorithm first finds the optimal d–packets to transmit as

$$Q_{ij}^{(d)}(n) = \max\left\{0,\ Q_i^{(d)}(n) - Q_j^{(d)}(n)\right\}, \quad (1)$$
$$d_{ij}^*(n) = \arg\max_d Q_{ij}^{(d)}(n).$$

To attribute a weight to each link, the HD algorithm performs the following:

$$\widehat{f_{ij}}(n) = \min\left\{\left\lceil 1/2\ Q_{ij}^{(d^*)}(n)\right\rceil,\ Q_i^{(d^*)}(n),\ \mu_{ij}(n)\right\}, \quad (2)$$
$$w_{ij}(n) = \left(\widehat{f_{ij}}(n)\right)\left(Q_{ij}^{(d^*)}(n)\right). \quad (3)$$

- **HD Scheduling:** After assigning the optimal weight (3) to each link, the scheduling matrix $\boldsymbol{S}(n)$ is chosen in a scheduling set $\boldsymbol{\mathcal{S}}$.
  It entails solving a centralized optimization problem:

  $$S(n) \in \arg\max_{s\in S} \sum_i S(e_i) w_{e_i},$$
  $$\text{subject to} \quad e_i, e_j \in S \Rightarrow \partial e_i \cap \partial e_j = \emptyset, \quad (4)$$

  where $i, j \in \mathcal{V}$ and $e_i, e_j \in \mathcal{E}$. $S(e_i)$ is a boolean variable taking value 0 or 1, with 1 denoting the edge e_i being activated in the scheduling set and 0 otherwise. w_{ij} is the edge weight. ∂e_i denotes the two nodes at both ends of the edge e_i.
- **HD Forwarding:** Subsequent to the scheduling stage, each activated link transmits $\widehat{f_{ij}}\,(n)$ number of packets in accordance with (2).

The Dirichlet protocol (a variant of HD) was originally proposed in [8]. The Dirichlet routing energy is defined as the sum of the squares of the link packet transmissions weighted by the link cost factors. It is proposed that the routing cost can be minimized if the routing protocol follows Dirichlet's Principle. The Dirichlet protocol is briefly reviewed as follows:

- **Dirichlet Weighting:** A set $K_{ij}(n)$ is defined such that $Q_{ij}^{(d)}(n) > 0, \forall d \in K_{ij}(n)$. Individual link cost is defined as ρ_{ij}.
  The Dirichlet weighting consists in solving the following optimization problem to find $\widehat{f_{ij}^{(d)}}(n)$:

  $$\text{Minimize} \sum_{d\in K_{ij}(n)} (\rho_{ij}^{(d)}(n)^{-1} Q_{ij}^{(d)}(n) - \widehat{f_{ij}^{(d)}}(n))^2 \quad (5)$$

  subject to

  $$\sum_{d\in K_{ij}(n)} \widehat{f_{ij}^{(d)}}(n) \le \mu_{ij}(n) \text{ and } 0 \le \widehat{f_{ij}^{(d)}}(n) \le Q_{ij}^{(d)}(n). \quad (6)$$

  Notations are similar to those in HD protocol. Then assign a weight to each class $d \in K_{ij}(n)$. The weight of each class d is given as

  $$w_{ij}^{(d)}(n) := 2\rho_{ij}^{(d)}(n)^{-1} Q_{ij}^{(d)}(n) \widehat{f_{ij}^{(d)}}(n) - (\widehat{f_{ij}^{(d)}}(n))^2 \quad (7)$$

  and the final link weight is

  $$w_{ij}(n) = \sum_{d\in K_{ij}(n)} w_{ij}^{(d)}(n). \quad (8)$$
- **Dirichlet Scheduling:** It is the same as the HD scheduling with 1-hop interference model.
- **Dirichlet Forwarding:** Subsequent to the scheduling stage, each activated link transmits a number $\widehat{f_{ij}^{(d)}}(n)$ of d packets.

Note that the main difference between Dirichlet routing and HD is in the weighting stage. In HD routing, half of the queue differential is considered for the most of the time while in Dirichlet, the full queue differential is in the weighting algorithm.

C. Ollivier-Ricci Curvature for Wireless Networks

The relatively new *Ollivier-Ricci curvature* [10], [11], [12] is a graph version of the well known Ricci curvature in differential geometry and, as already argued in [19], it is a fundamental tool in discrete heat calculus. It can be traced back to the Ricci curvature on manifolds providing an upper bound on the heat kernel [13, Sec. 3], [31, Sec. 5.6]. The Ollivier-Ricci curvature has already been used to anticipate congestion in a wireless network under the purely thermodynamical Heat Diffusion protocol [19]. The theoretical fact that underpins this observation is that the Ricci curvature regulates the flow of heat on a Riemannian manifold, in somewhat the same way that the sectional curvature regulates geodesics. Ollivier-Ricci curvature recent applications include formulating Hamiltonian Monte Carlo [27], unraveling Internet topology [30], modeling robustness of cancer networks [28], and analyzing market fragility and systemic risk [29]. However, here, our interest in the Ollivier-Ricci curvature rather stems from its definition in terms of a "transportation cost," which can be linked to queue occupancy, routing energy, even time to reach steady-state.

Consider a weighted graph $((\mathcal{V}, \mathcal{E}), \rho)$. On this graph, for each vertex i, we define a probability measure m_i on the neighboring nodes $\mathcal{N}(i)$ as follows:

$$\begin{aligned} m_i(j) &= \frac{\rho_{ij}}{\sum_{j\in\mathcal{N}(i)} \rho_{ij}}, \quad \text{if } ij \in \mathcal{E}, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

The Ollivier-Ricci curvature is defined in terms of the transport properties of the graph:

Definition 1: The Ollivier-Ricci curvature of the graph $((\mathcal{V}, \mathcal{E}), \rho)$ endowed with the set of probability measures $\{m_i : i \in \mathcal{V}\}$ is defined, along the optimal path $[i, j]$, as

$$\kappa([i, j]) = 1 - \frac{W_1(m_i, m_j)}{d(i, j)}, \quad (9)$$

where $W_1(m_i, m_j)$ is the first Wasserstein distance between the probability measures m_i and m_j defined on $\mathcal{N}(i)$ and $\mathcal{N}(j)$, resp.,

$$W_1(m_i, m_j) = \inf_{\xi^{ij}} \sum_{k,\ell\in\mathcal{N}(i)\times\mathcal{N}(j)} d(k, \ell)\xi^{ij}(k, \ell),$$

where the infimum is extended over all "coupling" measures ξ^{ij} defined on $\mathcal{N}(i) \times \mathcal{N}(j)$ and projecting on the first (second) factor as m_i (m_j), that is,

$$\sum_{\ell \in \mathcal{N}(j)} \xi^{ij}(k,\ell) = m_i(k), \quad \left(\sum_{k \in \mathcal{N}(i)} \xi^{ij}(k,\ell) = m_j(\ell) \right),$$

and $d(i,j)$ is the usual metric emanating from the edge weight ρ.

More intuitively, $\xi^{ij}(k,l)$ is called *transference plan.* It tells us how much of the mass of $k \in \mathcal{N}(i)$ is transferred to $l \in \mathcal{N}(j)$, but it does not tell us about the actual path that the mass has to follow.

The first Wasserstein distance is one class of shortest transportation distance between two probability distributions. For details of this concept, see [14], [15].

Bauer [12] developed a general sharp inequality for undirected, weighted, connected, finite (multi)graph of N vertices $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Note that it is computationally viable to solve the exact curvature instead of bounds. The optimal coupling can be solved via linear programming. The calculation of the 1^{st} Wasserstein distance is commonly referred to as *Earth Mover Distance* in Computer Science applications, such as pattern recognition.

III. Protocol Performance under Different Network Topology

A. Simulation Setup

The Back-Pressure, Heat Diffusion, Dirichlet, and Shortest Path Routing protocols were programmed and run in MATLAB 2015a. Link capacities are set to infinity unless otherwise specified. The centralized scheduling was implemented using Edmonds' blossom algorithm [9] and can be solved in $O(n^2m)$, where $n = |\mathcal{V}|$ and $m = |\mathcal{E}|$. However, the actual wall-clock time of MATLAB implementation of this algorithm is still extremely time-consuming in practice compared to other processes, and typically consumes more than 95% of processing time in every timeslot. For this reason, we propose, in [21], an Ising formulation of the scheduling problem under k-hop interference model for arbitrary k, which is proved to be NP-hard. The problems are solved efficiently on an adiabatic quantum computer (such as D-Wave).

All simulations are multi-class. Each node is sending packets to every other node. The packet arrival rate follows a Poisson distribution with $\lambda \in [2, 8]$. For the sake of the comparison analysis, Shortest Path Routing is assumed to be implementable in wireless sensor networks. Although it is not a fair comparison since Shortest Path Routing protocol (with global routing table) has access to more information than other protocols, the former is still being considered in order to show that the results are consistent with congestion results in wired networks in the sense developed in [3].

The Ollivier-Ricci curvature is calculated locally between each pair ∂e of nodes. Even though the Ollivier-Ricci curvature is a local concept, we nevertheless give it a global significance by taking its numerical average over all edges. This process is similar to the Gauss-Bonnet theorem where the local curvature is integrated to give a global Euler characteristic. Unfortunately, such a theorem for the Ollivier-Ricci curvature (rather than the sectional curvature) on discrete graphs has yet to be developed.

B. Node-to-Node Delay Performance

The average network delay is formulated as

$$\overline{Q} = \lim_{\tau \to \infty} \sup \frac{1}{\tau} \sum_{n=0}^{\tau-1} E\left\{ \sum_{i \in V} \sum_{d \in K} q_i^{(d)}(n) \right\}. \tag{10}$$

Since Poisson arrival rate is used in the simulation, by Little's Theorem, the expected time-averaged total queue congestion is proportional to the long-term averaged node-to-node network delay. Thus, it is sufficient to deal with average queue occupancy over all nodes in the network. As shown in Fig. 1, as the curvature of the underlying network becomes more positive, the node-to-node delay generally decreases in all four protocols being used in the comparison. It is worth noting the following:

- General Performance: Generally, the Heat Diffusion protocol performs worst in the sense of node-to-node delay; this is due to the unique packet forwarding mechanism, where only half of the queue differential is being transmitted. This feature is to prevent packet looping [7].
- Dirichlet vs. Shortest Path Routing: Although Shortest Path Routing has access to more information (global routing table), in more negatively curved networks, it still performs worse than Dirichlet routing in the sense of average delay, which is again a dynamic protocol where each node has information only on its neighboring nodes.
- Dirichlet routing energy: Note that the Dirichlet protocol proposed to minimize routing energy while ensuring queue stability for all stabilizable traffic [8]; however, in simulation it still entails relatively high routing energy. This is not a contradiction to the original theory. As shown in [19], protocols with higher steady state delay generally fails to stabilize faster as network capacity region slowly shrinks, and Dirichlet protocol has lowest queue occupancy among 3 dynamic protocols in comparison.

C. Routing Energy Performance

The total routing energy is formulated as

$$R(n) = \sum_{ij \in E} \sum_{d \in K} \rho_{ij} (f_{ij}^{(d)}(n))^2. \tag{11}$$

The routing energies of the four protocols under comparison are plotted in Fig. 2.

As shown the Dirichlet routing generally has higher total routing energy than other protocols, but performs relatively better than Back-Pressure in more negatively curved networks. It worth noting the following:

- General Performance: As the network is becoming more positively curved, the total routing energy decreases.

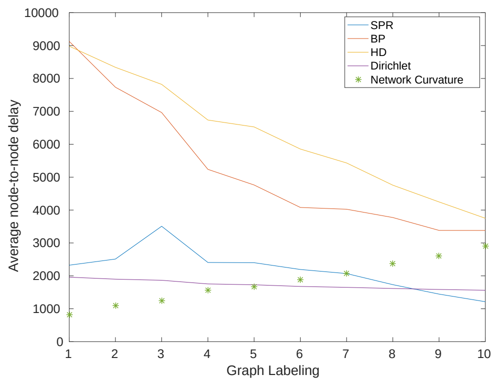

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Fig. 1: Steady state node-to-node delay of various routing protocols in different network topologies. SPR, BP, HD stands for Shortest Path Routing, Back Pressure, Heat Diffusion respectively

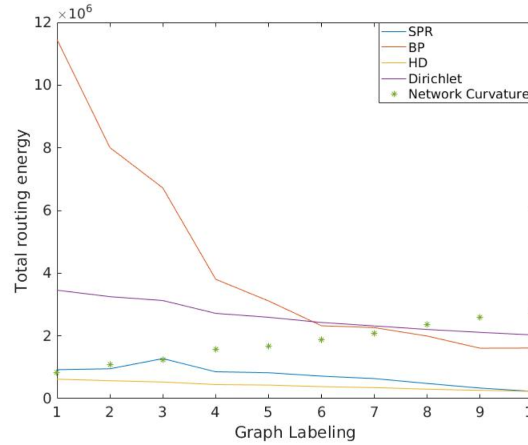

Fig. 2: Steady state routing energy of various routing protocols in different network topologies

Note that the routing energy is a summation measure instead of an average measure.

- Sensitivity to topology: As can be seen, the routing energy of the Back-Pressure protocol is much more sensitive to change of topology than other protocols. This will be analyzed in further details later.

D. Varying Topology and Sensitivity

Due to the nature of wireless sensor networks, the performance of protocols under changing topology is an important issue and a low sensitivity of delay and routing cost can be of value in some applications. Topology control in wireless networks also arises to cope with constantly changing network topology, such as the one in [16]. We believe that topology control can be achieved by curvature control in wireless networks. Curvature control has already been developed in wired networks [3], [5].

Here we would like to quantitatively analyze how varying topology would affect network performance. Simulation of varying topology is done by randomly deleting/adding nodes and links. The test graph is initialized as 30-node Erdös-Rényi graph with uniform edge weight. The simulation process is done in a controlled manner so that the overall curvature is decreasing for the first half of the process and then increasing for the second half, with details as follows:

- **Link addition and deletion:** At each timeslot, every pair of nodes in the global graph is retrieved. If no link is present between them, create a link with uniform probability and weighted as the initial weight. If a link already exists, delete it with same uniform probability. Deletion and addition will be performed exclusively for each timeslot to demonstrate the curvature change.
- **Node addition and deletion:** at each timeslot, certain nodes are to be deleted following uniform probability. If deleting such node results in disconnected graph, such deletion is forbidden. Thus, the rank of the Laplacian matrix has to be checked every time before deletion. Certain nodes are to be added and linked to be established to other nodes following uniform probability as well.
- **Weight variation:** Weight variation is done by putting a new weight to every existing link at every timeslot, following a Gaussian distribution.

By adding more links to the graph, deleting nodes with lower degree, adding more nodes to the centroid of the graph, the curvature generally becomes more positive, and vice versa. As shown in Fig. 3, different routing protocols performs significantly differently in varying topologies, but interestingly, both the delay and the routing energy of the Dirichlet protocol remain in a very steady state situation throughout the process of varying of the curvature.

Sensitivity Analysis:
As can be seen from Figs. 1 and 3, the node-to-node delay and the routing energy significantly increase as the topology of the network becomes more negative in Shortest Path Routing, Back Pressure and Heat Diffusion, but remains relatively stable for the Dirichlet protocol. We define measures of protocol sensitivity to topology variation as

$$S_Q = \left|\frac{dQ_{\text{average}}}{d\kappa_{\text{average}}}\right|, \quad S_R = \left|\frac{dR_{\text{average}}}{d\kappa_{\text{average}}}\right|, \qquad (12)$$

that is, the sensitivity of a protocol node-to-node delay and routing energy, respectively, to curvature variation. Q_{average} denotes the total queue occupancy averaged over 20 timeslots

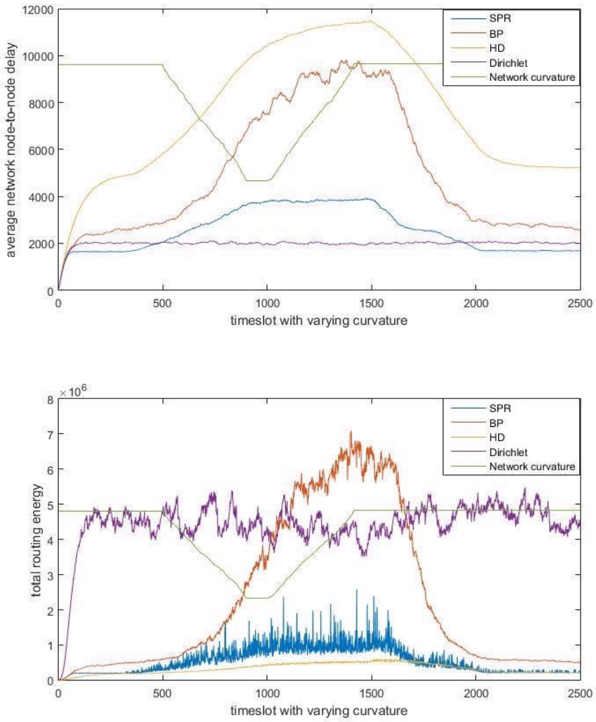

Fig. 3: Performance and sensitivity of different routing protocols in varying topology

TABLE I: Numerical values of sensitivity of four protocols. S_Q is in the scale of 10^3, S_R is in the scale of 10^7.

	S_Q	S_R
Shortest Path	2.40	0.78
Backpressure	9.01	1.14
Heat Diffusion	5.25	0.05
Dirichlet	1.48	1.21

(rather than over infinitely many timeslots as in Eq. (10)) to smooth over the transients of the network dynamics; R_{average} denotes the total routing energy (11) averaged over 20 time slots; and κ_{average} is the Ollivier-Ricci curvature averaged over 20 timeslots. We propose to use them as metrics for network routing protocol. Also $\frac{1}{S}$ could be used to denote the protocol robustness to changing topology. The numerical results of the sensitivity of different protocols is summarized in Table I. The Dirichlet protocol has lowest sensitivity, thus highest robustness to varying topology in the sense of node-to-node delay, while Heat Diffusion has highest robustness in the sense of routing energy. Subject to particular application needs, different protocols can be chosen selected according to their sensitivity metrics.

E. Summary of Topological Impacts

It has been shown via several numerical examples that the performance of different routing protocols varies significantly across networks with different topological characteristics. The Dirichlet and Back-Pressure protocols generally consume more routing energy than the Shortest Path Routing and Heat Diffusion protocols, but keep delay reasonably low. The Heat Diffusion protocol, on the other hand, consumes as less routing energy as possible, while creating much higher delay. Its routing energy is both the lowest and the least sensitive to changing topology, while its delay is both the highest and the most sensitive. This is believed to be a result of the laziness feature that is developed against packet looping.

Note that compared to Dirichlet routing and Heat Diffusion, Back-Pressure does not have an apparent strength in terms of delay or routing energy, but it rather performs in a mediocre manner on both metrics. This illustrates again the delay-energy tradeoff discussed in [18]. Thus, different protocols can be chosen according to different application needs, or a control approach can be developed to switch routing protocol in a graph curvature feedback scheme.

IV. Curvature-driven adaptive control

It is sometimes desirable for a network to be capable of multi-protocol switching, such as the RFID reader system in [17], especially under potentially significant curvature change that could degrade network performance below specifications. We hereby propose an adaptive control scheme taking switching decisions on the base of network curvature identification data to keep network performance as desired. Fig. 4 shows the overall architecture of the adaptive protocol. Instead of an adaptation law depending on a *numerical* estimate of the curvature, we propose a decision tree adaptation law that utilizes the *sign* of the curvature only, as shown in Fig. 5. The switching decision process can be done either every timeslot or preferably only once over an arbitrary number of timeslots since topology variation is mostly likely much slower compared with timeslot.

Also, it is important to note that the implementation of such a control scheme requires a global gateway that sends the information of global curvature to every node. Since Ollivier-Ricci curvature is by itself a local measure, it is worth investigating in the future how control based on local curvature would affect the global performance by assuming that each node could operate under its own protocols in some synchronous manner.

V. Conclusion and Future Work

Following in the footsteps of our previous work [19], we have further developed the quantitative analysis of different protocols under varying topologies and proposed a possible adaptive control method driven by a global curvature metric. Relations between congestion and network topology are consistent in both wired and wireless networks, except that the former one uses the Gromov concept whereas the latter uses the more recent Ollivier-Ricci curvature. It is also worth

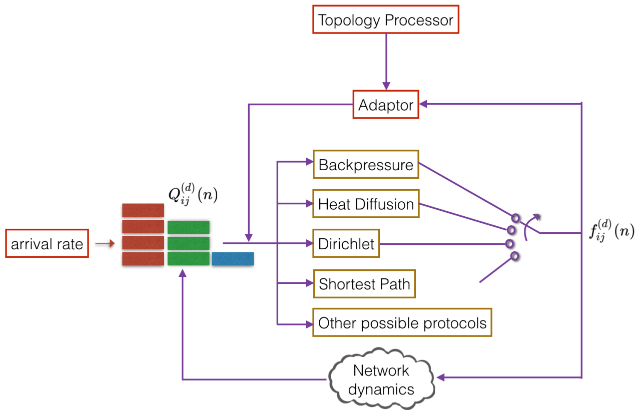

Fig. 4: Adaptive system switching to best protocol given current topology and desired performance metric

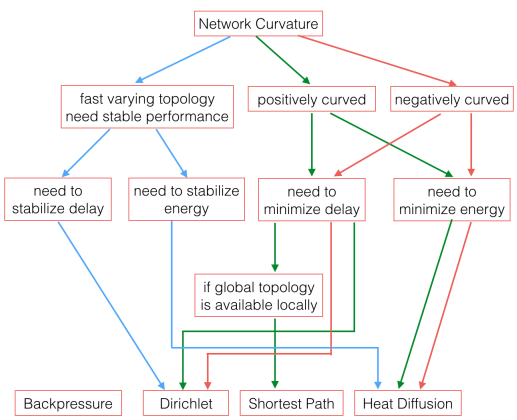

Fig. 5: Decision tree based switching logic of the adaptive system of Fig. 4. Each arrow color represents different categorization of curvature.

noting that, in negatively curved networks, if every node is acting in its own interest, it would deteriorate the overall performance in both wired and wireless networks. This could have potential applications in social network studies and game theory. In the near future, we would implement the control in more practical setups and further develop potential control and decision making methods based on local Ollivier-Ricci curvature and thus further develop protocol based on topology and curvature control.

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