Applying classical control techniques to quantum systems: entanglement versus stability margin and other limitations

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Outline

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- The system: two qubits in a lossy cavity
- The performance measures considered here
- Bounding the steady-state error
- Concordance/discordance of performance measures
- Conclusions

Motivation

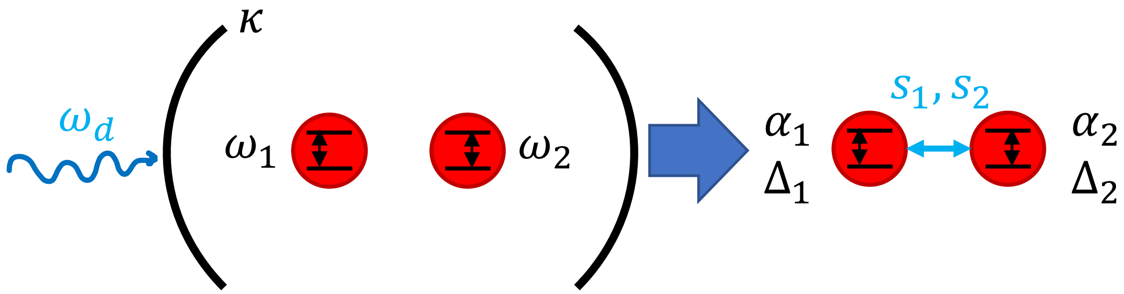
- Classical robust control theory is well-established, but it is marginally successful when applied to quantum mechanical systems
- Why?
  - For a closed (i.e., unitary) quantum system, the poles are purely imaginary
    - $\partial_t|\psi\rangle = -iH|\psi\rangle$
  - Open quantum systems subject to decoherence and dissipation will have eigenvalues with negative real parts
    - $\partial_t\rho = -i[H,\rho] + \sum_k \left(V_k^\dagger \rho V_k - \frac{1}{2}\{V_k V_k^\dagger, \rho\}\right)$
    - These systems still retain a zero eigenvalue due to trace constraints $Tr\{\rho\} = 1$
    - Risk of a loss of “quantumness” if dissipative effects are too strong

Motivation

- Classical robust control theory is well-established, but it is marginally successful when applied to quantum mechanical systems
- Why?
  - Many quantum performance measures are nonlinear, e.g., entanglement measures, squeezing measures
- Most of the time, physicists simply try to determine robustness via Monte-Carlo sampling, but we should try to find the fundamental limits imposed by control theory.
- The goal of this work is to motivate the need for a general theory of robust quantum control

The system: two qubits in a lossy cavity

- Inspired by F. Motzoi, et al. Phys. Rev. A **94**, 032313, (2016).
    - Qubit transition frequency ω_1, ω_2
    - Cavity resonance frequency ω_d (detuning $\Delta_\ell = \omega_\ell - \omega_d, \ell = 1{,}2$)
    - Qubit driving amplitude a_ℓ
    - Cavity coupling κ
- Can perform a unitary transformation to adiabatically eliminate the cavity
    - Cavity remains via a dissipative collective qubit coupling term that takes the form $V_c = s_1\sigma_1^- + s_2\sigma_2^-$



The system: two qubits in a lossy cavity

- System evolution governed by the Lindblad master equation (4x4 density matrix ρ)

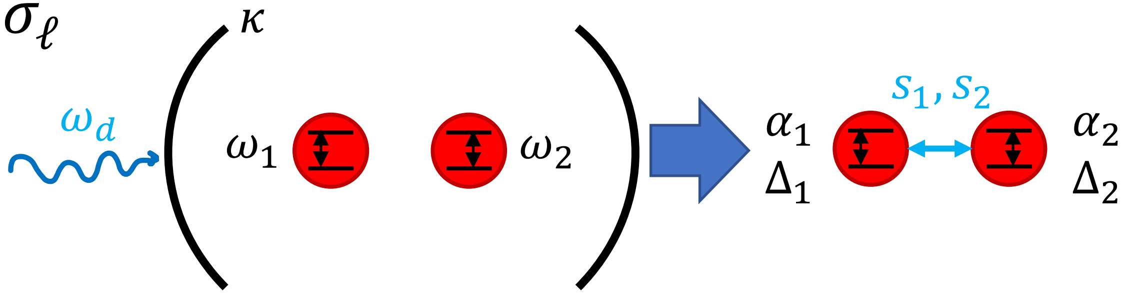
$$\partial_t \rho = -i[H,\rho] + \sum_k \left(V_k^\dagger \rho V_k - \frac{1}{2}\{V_k V_k^\dagger, \rho\} \right)$$

$$H = \sum_\ell \alpha_l \sigma_\ell^+ + \alpha_\ell^* \sigma_\ell^- + \Delta_\ell \sigma_\ell^+ \sigma_\ell^-$$

$$V_c = s_1 \sigma_1^- + s_2 \sigma_2^-$$

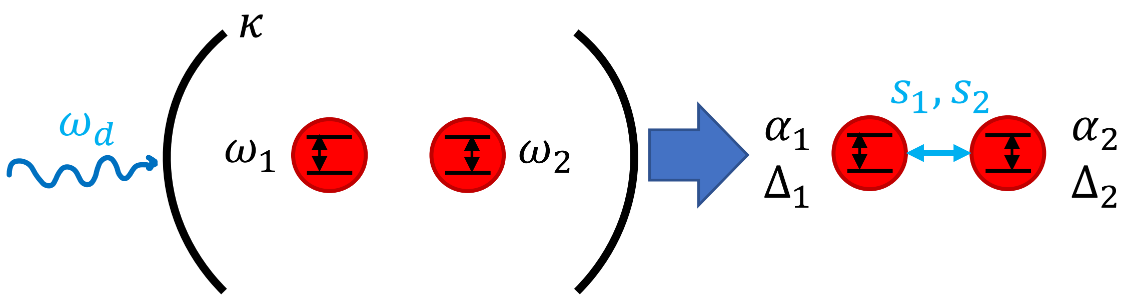
$$V_{(\ell,\phi)} = \gamma_\ell^{(\phi)} \sigma_\ell^{(z)}$$

$$V_{(\ell,r)} = \gamma_\ell^{(r)} \sigma_\ell^-$$



The system: two qubits in a lossy cavity

- Solve this in the Bloch representation where the density matrices are represented by 16-element vectors $\boldsymbol{r}$ corresponding to coefficients of a basis of 15 traceless Hermitian operators on a Hilbert space with dimension 4 (and the identity operator) $\{\nu_k\}$
- Obtain a 16x16 matrix $\boldsymbol{A}$ describing the system dynamics with $\dot{\boldsymbol{r}} = \boldsymbol{A}\boldsymbol{r}$
- It is straightforward to go between ρ and $\boldsymbol{r}$, but a global steady-state is easily found by setting $\dot{\boldsymbol{r}} = 0$.



Perturbations

- Define a bare (unperturbed) steady-state $\rho_{ss}(0)$ and a perturbed steady-state $\rho_{ss}(\delta)$
  - Denote our perturbation magnitude with δ
- Consider a system with bare parameters

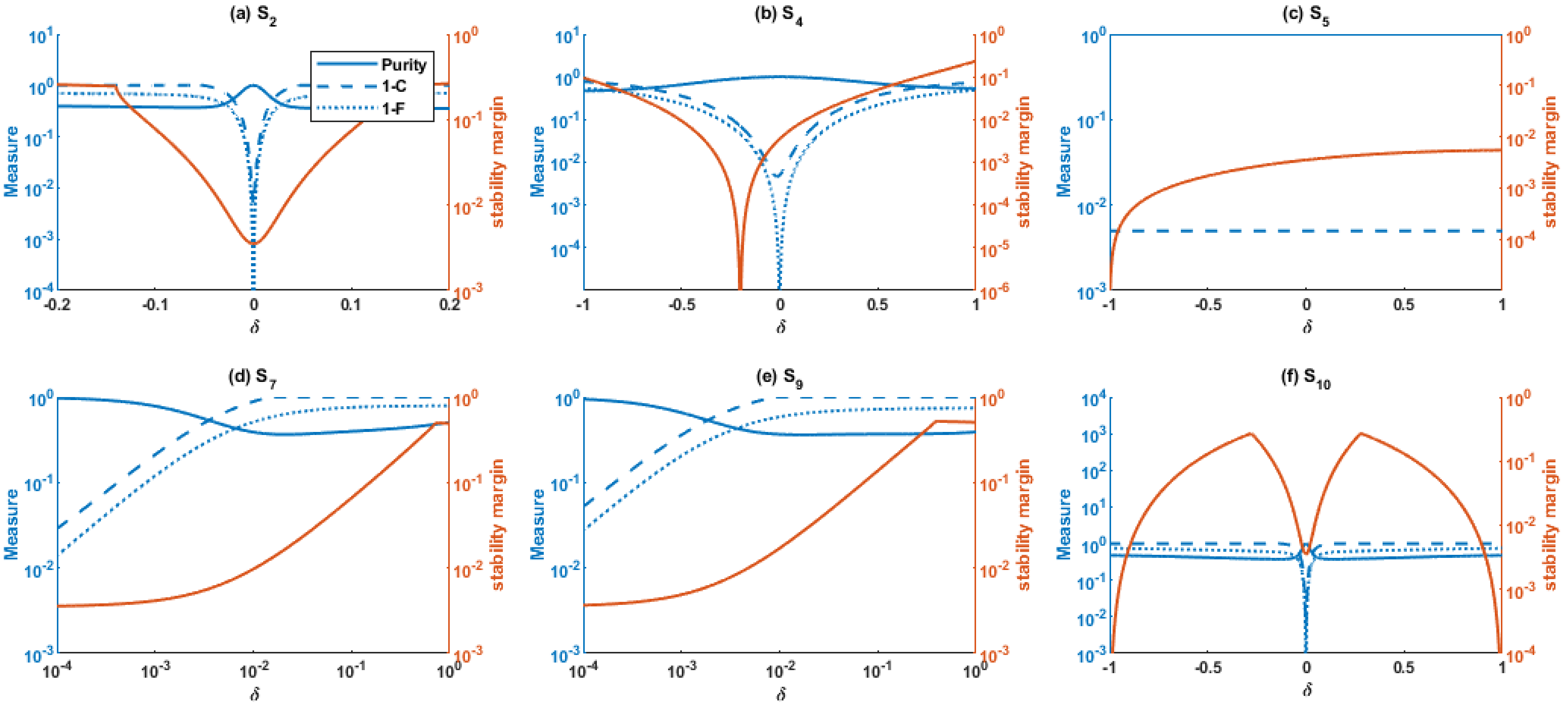
$$\left(\alpha_1, \alpha_2, \Delta_1, \Delta_2, s_1, s_2, \gamma_1^{(\phi)}, \gamma_2^{(\phi)}, \gamma_1^{(r)}, \gamma_2^{(r)}\right) = (1, 1, 0.1, -0.1,\ 1, 1, 0, 0, 0, 0)$$

- Perturbations (notation consistent with S.G. Schirmer et al, IEEE TAC **67**, 11, (2022)):
  - $S_2 = \{0,1,0,0,0,0,0,0,0,0\}$ (perturbation to qubit 2 driving)
  - $S_4 = \{0,0,0,1,0,0,0,0,0,0\}$ (perturbation to qubit 2 detuning)
  - $S_5 = \{0,0,0,0,1,1,0,0,0,0\}$ (symmetric perturbation to collective coupling)
  - $S_{10} = \{0,0,0,0,1,-1,0,0,0,0\}$ (anti-symmetric perturbation to collective coupling)
  - $S_7 = \{0,0,0,0,0,0,0,1,0,0\}$ (perturbation to qubit 2 decay)
  - $S_9 = \{0,0,0,0,0,0,0,0,0,1\}$ (perturbation to qubit 2 dephasing)

Performance measures (at steady-state)

- Here, we consider the following performance measures:
  - Concurrence $\tilde{C}(\delta) \in [0,1]$
    - $\tilde{C}(\delta) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$ where the λ_k are the ranked eigenvalues of the matrix $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$, $\tilde{\rho} = \left(\sigma^{(y)} \otimes \sigma^{(y)}\right)\rho^*\left(\sigma^{(y)} \otimes \sigma^{(y)}\right)$ and $\sigma^{(y)} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ is the Pauli y-matrix for a given $\rho = \rho_{ss}(\delta)$.
    - This is a measure of entanglement, where $\tilde{C}(\rho) = 1$ for a fully entangled state (e.g. a Bell state)
    - $C(0) = 0.995$
  - Fidelity $F \in [0,1]$ of $\rho_{ss}(\delta)$ relative to $\rho_{ss}(0)$
    - $F(\delta) = Tr\{\rho_{ss}(0)\rho_{ss}(\delta)\}$
  - State purity $Tr\{\rho_{ss}^2\} \in [0,1]$, purity of $\rho_{ss}(0) = 1$
  - Classically inspired stability margin $G(\delta) = |\max(\mathcal{R}\{\lambda_n(\delta)\})|$ for the eigenvalues of $\boldsymbol{A}$ corresponding to a given steady state $\rho_{ss}(\delta)$ (ignoring the zero eigenvalue)
    - $G(0) \approx 0.01$

Results: perturbations to $\rho_{ss}(0)$



Bounding the steady-state error

- In general, the Bloch matrix and structured perturbation matrix take the form

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ 0 & 0 \end{bmatrix}, \boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} \\ 0 & 0 \end{bmatrix}$$

- Given a perturbation δ, can bound the steady-state error as

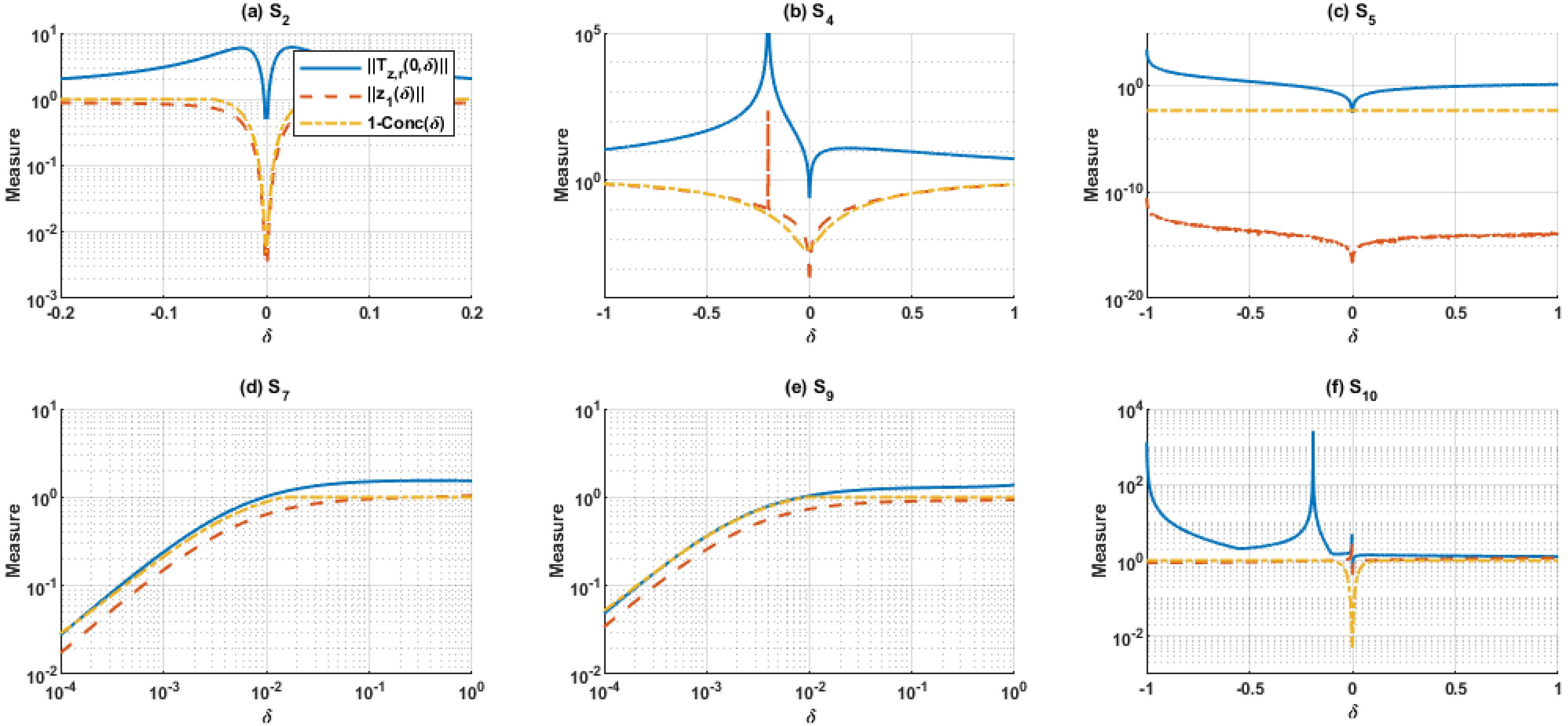
$$\|\boldsymbol{z}(\delta)\| = \|\boldsymbol{r}_{ss}(\delta) - \boldsymbol{r}_{ss}(0)\| = \lim_{s\to 0}\left\|\boldsymbol{T}_{\boldsymbol{z},\boldsymbol{r}}(\delta, s)s\hat{\boldsymbol{r}}(s)\right\| \leq \left\|\boldsymbol{T}_{\boldsymbol{z},\boldsymbol{r}}(s, 0)\right\|\|\boldsymbol{d}\|,$$

where

$$\boldsymbol{d} = \lim_{s\to 0} s\hat{\boldsymbol{r}}(s) = [\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}; 1]/N,$$

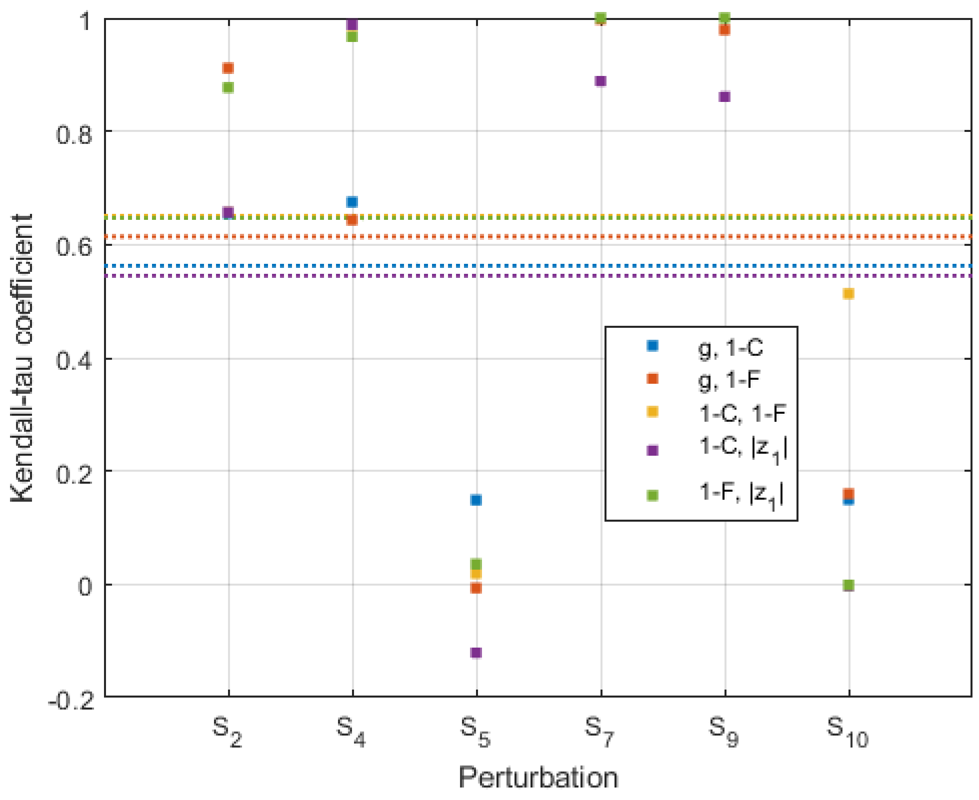
and $\boldsymbol{T}_{z,r}(\delta, s)$ is the transfer function that takes us from $\hat{\boldsymbol{r}}(s)$ to $\hat{\boldsymbol{z}}(s)$.

Results: bounding the steady-state error



Concordance/discordance, Kendall-tau analysis

- To understand the concordance/ discordance of these parameters, we ran Kendall-tau analyses between them
- While there is general concordance between the different measures, this is not universally true
- Obviates the need for more detailed work and a general theory!



Conclusions

- These results show that a more general theory of quantum robust control is necessary
- Recent work of ours has shown the efficacy of a robustness infidelity measure (RIM) in determining the robustness of quantum controllers (arXiv:2207:07801)
- We have also shown recently that a time-domain version of the log-sensitivity has utility in quantum problems, both unitary and dissipative (arXiv:2210.15783)
- More work remains to be done!
- There is a need to bridge the gap between the control and physics communities