Reversible Models for Wireless Multi-Channel Multiple Access

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***Abstract*—This paper presents a network layer model for a wireless multiple access system with both persistent and nonpersistent users. There is a single access point with multiple identical channels. Each user who wants to send a file first scans a subset of the channels to find one that is idle. If at least one idle channel is found, the user transmits a file over that channel. If no idle channel is found, a persistent user will repeat the access attempt at a later time, while a nonpersistent user will leave. This is a useful mathematical model for situations where a group of persistent users stay near an access point for an extended period of time while nonpersistent users come and go. Users have heterogeneous activity behavior, file upload rates, and service durations. The system is a complex multi-dimensional Markov chain. The steady state probabilities are found by exploiting a latent reversibility property and leveraging a discrete Fourier transform. This enables simple expressions for throughput and blocking probability.**

I. Introduction

Modern wireless multiple access networks provide a broad frequency spectrum that can be divided into a large number of channels. These channels can be used to simultaneously support traffic from multiple users. However, it is not trivial to keep track of the idle/busy state of each channel. This problem is even more difficult when users have dynamic behavior. For example, in internet-of-things applications, users may arrive randomly, send a small burst of data, and then leave. There is no time to coordinate a channel-sharing schedule. Further, scanning the state of all channels before transmission can incur a large time and energy overhead [1]. Wideband techniques for sensing many channels simultaneously are treated in [2] [3] [4]. Narrowband techniques for sensing one channel at a time are in [5] [6] [7] [8] [9] [10] [11] [12]. Multi-channel scanning for fast contention resolution is explored in [13] [14] [15].

This paper considers a simple network model of multi-channel multiple access and studies the resulting network dynamics. We assume there are m identical channels. To reduce the time and energy cost of spectrum sensing, we assume each user randomly scans only a *subset* of s channels, where $1 \leq s \leq m$. The probability of finding an available channel depends on s and on the number of channels that are currently busy. The value of s can be set according to the particular narrowband (small s) or wideband (large s) sensing techniques used, leveraging work such as in [1]- [12]. Even under this simple model, the network state dynamics are complex. Users are heterogeneous, can be in different activity states, and can have different file size parameters. The network state includes which users are active and what activity states they are in. The state space is ginormous: It grows exponentially with the number of users and can easily be larger than the number of atoms in the galaxy. It is important to develop simple mathematical models for these complex networks.

This paper considers a continuous time Markov chain model for these networks. While the number of states is very large, the model has a hidden *reversibility* property that enables exact computation of the steady state probabilities. To develop the method, Section II first treats a simple situation where all users are *nonpersistent*, meaning that each user arrives and has only one file to send before departing. This situation allows a simple expression for network throughput that holds for multiple classes of nonpersistent users, each class having different file size and arrival rate parameters.

A more complex scenario with both persistent users and nonpersistent users is treated in Section III. A persistent user is a user that remains close to the wireless access point for a long duration of time, repeatedly sends many files, but has random activity patterns described by a simple 3-state diagram (see Fig. 1). The activity states for each user can have different transition rate parameters. Fortunately, the same reversibility technique can be used to compute the exact steady state probabilities. Unfortunately, in this scenario it is not clear how to sum over the (overwhelmingly large) number of states to compute throughput and blocking probabilities. This is a known challenge of reversible systems in other contexts. Indeed, in reversible networks of truncated $M/M/\infty$ queues, it can be shown that steady state probabilities can be computed up to a scaling constant B, but calculating B to within a reasonable approximation can be NP-hard in general [16] (see also [17] for factor graph approximation methods). It is not obvious if the model of the current paper admits efficient computation. This was a significant challenge in the development of this work. Fortunately, this paper overcomes this challenge by showing the network model has structure that admits computation of the desired throughput and blocking probabilities in polynomial time. Our solution carefully sums over all (exponentially many) probabilities by invoking a transform domain argument via a discrete Fourier transform.

A. Network model

Consider a wireless system with a single access point that has m identical channels, where m is a positive integer. Each channel can support one file transmission, so that up to m files can be transmitted simultaneously. Different types of users want to upload files to the access point. To do this, they first

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need to find an idle channel. At the start of every upload attempt, a user randomly scans a subset of the channels in hopes of finding at least one channel that is idle. Let s be the size of the subset that is scanned and assume that $1 \leq s \leq m$. Example numbers are $m = 50$ and $s = 5$, so that every user randomly scans 5 of the 50 channels. If an idle channel is found, the user sends a single file over that channel (if multiple idle channels are found, the choice of which one to use is made arbitrarily). If no idle channel is found, the users react differently depending on their type: Persistent users try again later, while nonpersistent users leave and do not return.

An example of this situation is when the access point is in a fixed location, such as in a coffee shop. Persistent users are customers who find a table at the coffee shop, stay for an extended period of time, and use their wireless devices during their stay. Non-persistent users either walk past the coffee shop without entering, or enter only for a short time (perhaps to place a take-out order).

The system operates in continuous time over the timeline $t \geq 0$. A channel $j \in \{1, \ldots, m\}$ is said to be *busy* at time t if it is currently being used, that is, if there is a user that is transmitting a file over that channel. Let $B(t) \in \{0, 1, \ldots, m\}$ be the total number of channels that are busy at time t. Suppose a user who is not currently transmitting attempts to access the network at a time t for which $B(t) = b$. This user scans a random subset of s channels to determine which (if any) are not being used, with all subsets equally likely. Let $\theta(b)$ denote the conditional probability that a user successfully finds an idle channel, given that $B(t) = b$. We assume only that $\theta(b)$ satisfies the following basic properties:

$$0 \leq \theta(b) \leq 1 \quad \forall b \in \{0, 1, 2, \ldots, m\} \tag{1}$$

$$\theta(b) > 0 \quad \forall b \in \{0, 1, 2, \ldots, m-1\} \tag{2}$$

$$\theta(m) = 0 \tag{3}$$

Requirement (1) ensures $\theta(b)$ is a valid probability for each $b \in \{0, 1, 2, \ldots, m\}$; requirement (2) ensures that it is possible to utilize all m channels simultaneously; requirement (3) enforces the physical constraint that the system cannot support more than m active channels simultaneously.

For example, if $s = m$, so that all channels are scanned, then we have $\theta(b) = 1$ if $b < m$ and $\theta(m) = 0$. This is a nontrivial special case because the network dynamics are still complex in this case. However, it is useful to use smaller values of s to reduce the time and energy required for scanning. If $s = 1$, so that only one channel is scanned, and if this channel is independently and uniformly selected over all m channels, then $\theta(b) = 1 - b/m$ for all $b \in \{0, 1, \ldots, m\}$. More generally, if $s \in \{1, \ldots, m\}$, and a random subset of s distinct channels is scanned independently and uniformly over all subsets of size s, it can be shown that:

$$\theta(b) = \begin{cases} 1 & \text{if } b \in \{0, 1, \ldots, s-1\} \\ 1 - \left(\frac{b}{m}\right)\left(\frac{b-1}{m-1}\right) \cdots \left(\frac{b-(s-1)}{m-(s-1)}\right) & \text{if } b \in \{s, \ldots, m\} \end{cases} \tag{4}$$

An interesting feature of this model is that the probability of successfully finding an idle channel at time t depends only on $B(t)$, not on which types of users are using each channel. However, $B(t)$ is not enough to describe the state of the system: The full system state is described by a multi-dimensional Continuous Time Markov Chain (CTMC).

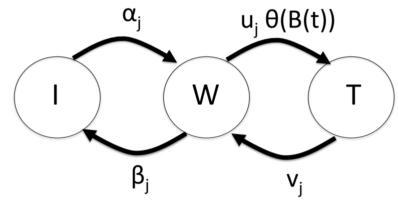

Fig. 1. A state diagram showing the idle, waiting, and transmitting states for persistent user $j \in \{1, \ldots, n\}$.

Section II considers the case when all users are *nonpersistent*, meaning that each user arrives and attempts to send one file before it leaves. Section III treats the more complex case with both persistent and nonpersistent users. Each persistent user $j \in \{1, \ldots, n\}$ has a 3-state activity process with *idle*, *waiting*, and *transmitting* states (see Fig. 1). The transition rates between the three states can be different for each user. It is important to emphasize that the 3-state process shown for a single persistent user j in Fig. 1 is not a complete description of a Markov chain. That is because the transition rate for the $W \to T$ transition is $u_j\theta(B(t))$, a multiplication of the user j attempt rate u_j with the success probability $\theta(B(t))$. Here, $B(t)$ is the *time-varying* number of busy channels. Therefore, the user-j transition rate for $W \to T$ itself depends on the state of all other users.

B. Related work

Our system model is similar to recent work in [18] that also treats multi-channel systems that scan a subset of the m available channels before transmission. The work in [18] also assumes that users are in one of three states (idle, probing, transmitting), which is similar to our 3-state persistent user structure. The work in [18] does not solve the resulting steady state probabilities, rather, it develops mean-field results that are asymptotically accurate when all users have identical parameters and when the network size scales to infinity, in which case a limiting ordinary differential equation (ODE) governs large scale behavior. Related mean field analysis and limiting ODE analysis are considered for Aloha-type systems in [19] and CSMA-type systems in [20], again assuming a large system limit and homogeneous conditions. In contrast, our work provides the exact steady state values for the continuous time Markov chain for any system size and for heterogeneous user parameters. It also treats the case when both persistent and nonpersistent users are present. It should be emphasized that our work exploits a latent reversibility property that does not exist in the model of [18]. In particular the 3-state user dynamics of [18] can roughly be viewed as similar to those of Fig. 1 with the exception that there is no $W \to I$ transition, and the $T \to W$ transition is replaced with a $T \to I$ transition. Intuitively it is clear that if it is possible to have a transition $I \to W$ but impossible to have a transition in the opposite direction, then reversibility fails. It is not clear if exact steady state behavior can be obtained when reversibility fails; that remains an important open question and mean field analysis is a central technique for those situations.

Recent work that uses a distributed multi-armed bandit approach to drive users towards a utility optimal configuration is in [21] and a related regret analysis for a distributed 2-user protocol is in [22]. A general theory of reversibility in Markov chains is described in [23] and reversibility for truncated $M/M/\infty$ systems is in [24] (see also [16] [17]). The Markov chain we consider in Section II has a structure similar to (but not the same as) the *open migration processes* described in [23].

Our model of persistent users accounts for heterogeneous *human user activity*, where users can be in various states depending on their activity patterns. The topic of mathematical models for human-based activity patterns for wireless communication is of recent interest. For example, related Markov-based models of human user activity and human response times are treated in [25] for wireless scheduling; related 2-state user activity models are used in [26] to treat file downloading as a constrained restless bandit problem.

C. Our contributions

The first main contribution is the presentation of a mathematical model for the dynamics of multi-channel wireless networks. The model allows users to have heterogenous file size and arrival parameters, which is useful for treating human-based activity patterns. The model captures the ginormous state space of the network. However, our analysis leverages a hidden reversibility property that, we show, allows exact analysis of the steady state probabilities. The second main contribution is our polynomial-time computation formula that leverages Fourier theory to sum over an exponentially-large number of probabilities to obtain individual blocking probability and throughput for each user.

II. Non-persistent users

This section considers the simple case where all users are nonpersistent. Each user arrives once and makes one attempt to access a channel. If the access is successful then the user transmits its file, else it leaves and does not return. Assume there are k classes of such users, where k is a positive integer. Users from each class $i \in \{1, ..., k\}$ arrive according to independent Poisson processes with rates $\lambda_1, \dots, \lambda_k$. Each user has one file to send. File service times are independent. Files of class i have service times that are exponentially distributed with parameter μ_i. Assume that $\lambda_i > 0$ and $\mu_i > 0$ for all $i \in \{1, \dots, k\}$. The different classes can be used to represent different communities of users who may have different arrival rate and file size parameters.

A. Markov chain model

The system can be modeled as a continuous time Markov chain (CTMC) with vector state $X(t) = (X_1(t), ..., X_k(t))$, where $X_i(t)$ is the number of type i files currently transmitting at time t. The state space $\mathcal{S}$ is given by the set of all vectors $(x_1, ..., x_k)$ that have nonnegative integer components such that $\sum_{i=1}^k x_i \leq m$, where m is the number of channels (assume m is a positive integer). Let $B(t) = \sum_{i=1}^n X_i(t)$ be the number of busy channels. A user that arrives to the system scans a subset of the channels to find one that is idle. For each $b \in \{0, 1, ..., m\}$ define $\theta(b)$ as the conditional probability that a newly arriving user finds an available channel, given that $B(t) = b$. The $\theta(b)$ values are assumed to satisfy (1)-(3).

To completely describe the Markov chain structure of this system, it remains to specify the transition rates. The transition rates $q_{w,z}$ between two states $w = (x_1, ..., x_k)$ and $z = (y_1, ..., y_k)$ are as follows: Fix an integer $j \in \{1, ..., k\}$ and define $e_j = (0, 0, ..., 0, 1, 0, ..., 0)$ as the vector that is 1 in entry j and zero in all other entries. Let $x = (x_1, ..., x_k)$ and $x + e_j = (x_1, ..., x_j + 1, .., x_k)$ be two states in the state space $\mathcal{S}$. Then

- Transition rate $x \to x + e_j$ is given by
$$q_{x,x+e_j} = \lambda_j \theta\left(\sum_{i=1}^k x_i\right)$$
This is the product of the arrival rate λ_j with the success probability given that the new user scans when the system state is $x = (x_1, ..., x_k)$.
- Transition rate $x + e_j \to x$ is given by
$$q_{x+e_j,x} = (x_j + 1)\mu_j$$
This is because there are currently $(x_j + 1)$ jobs of type j that are actively using channels, and each has an exponential service rate equal to μ_j.

Since the system state can change by at most one at any instant of time, there are no other types of transitions and so $q_{w,z} = 0$ for states $w, z \in \mathcal{S}$ that do not have the above form. It is not difficult to see that the Markov chain is *irreducible*, so that it is possible to get from any state of the state space $\mathcal{S}$ to any other state in $\mathcal{S}$ (the requirement (2) and the fact that $\lambda_i > 0$ for all $i \in \{1, \dots, k\}$ ensure this).

B. Basic Markov chain theory

This subsection recalls basic Markov chain theory (see, for example, [23] [27] [24]). Consider a continuous time Markov chain (CTMC) with a finite or countably infinite state space $\mathcal{S}$ and transition rates $q_{w,z} \geq 0$ for all $w, z \in \mathcal{S}$. Assume that $q_{w,w} = 0$ for all $w \in \mathcal{S}$. The states of $\mathcal{S}$ can be viewed as nodes of a graph; the links of the graph are defined by state-pairs (w, z) such that $q_{w,z} > 0$; the CTMC is said to be *irreducible* if this graph has a path from every node to every other node. A *probability mass function* over the state space $\mathcal{S}$ is a vector $(p(w))_{w \in \mathcal{S}}$ that satisfies $p(w) \geq 0$ for all $w \in \mathcal{S}$ and $\sum_{w \in \mathcal{S}} p(w) = 1$. The goal is to find a mass function that satisfies the following *global balance equations*:

$$p(w) \sum_{z \in \mathcal{S}} q_{w,z} = \sum_{z \in \mathcal{S}} p(z) q_{z,w} \quad \forall w \in \mathcal{S} \tag{5}$$

It is well known that if the CTMC is irreducible and has a finite state space, then there is exactly one probability mass function $(p(w))_{w \in \mathcal{S}}$ that solves (5), and this is the steady state mass function.

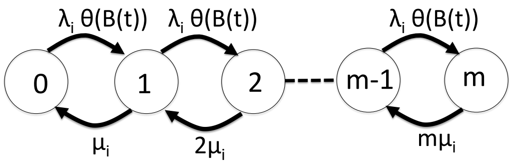

Fig. 2. A birth-death-like state diagram of a single user class $i \in \{1, \ldots, k\}$.

An irreducible CTMC is said to be *reversible* if there exists a probability mass function $(p(w))_{w\in\mathcal{S}}$ that satisfies the following *detailed balance equations*:

$$p(w)q_{w,z} = p(z)q_{z,w} \quad \forall w, z \in \mathcal{S} \tag{6}$$

It is well known that if a probability mass function $(p(w))_{w\in\mathcal{S}}$ solves the detailed balance equations, then it also satisfies the global balance equations and hence is the unique steady state. However, not all CTMCs are reversible. That is, not all CTMCs have steady states that satisfy (6).

C. Steady state for nonpersistent users

It is not immediately apparent whether or not the Markov chain for our system of nonpersistent users is reversible. Fortunately, the system is similar to an *open migration process* with reversibility properties as described in [23]. An open migration process is a system with k *colonies* that can be described by a Markov chain of the type $(X_1(t), ..., X_k(t))$, where $X_i(t)$ is the current population of colony i, transitions between states occur when a single member of colony i moves to colony j, and transition rates for such events depend only on the current population X_i. Such a migration process can *almost* be used to model the multi-access system of interest because the number of type i jobs currently using channels can intuitively be viewed as the population of "colony i." However, the multi-access system is not a migration system because the transition structure is different and transition rates depend on the *sum* population $x_1 + ... + x_k$. Nevertheless, reversibility properties can be established. To this end, define

$$\rho_i = \lambda_i/\mu_i \quad \forall i \in \{1, ..., k\}$$
$$\rho = \sum_{i=1}^{k} \rho_i$$

The technique behind the next theorem is to *guess* a probability mass function and then show the guess satisfies (6). The structure of the guess is not obvious. However, to gain intuition, note that we constructed our guess for the probability of state $(x_1, \ldots, x_k)$ by observing the "birth-death-like" structure of the system in Fig. 2 and guessing that steady state is a product of terms that include factors of the type $\rho_i^{x_i}/x_i!$ (which are also factors in the steady state mass function of a 1-dimensional $M/M/\infty$ queue) as well as factors that multiply the chain of success probabilities $\theta(r)$ over all $r \in \{0, \ldots, x_1 + ... + x_k - 1\}$. Once a good guess is made, it is not difficult to verify the guess satisfies the detailed balance equations.

Theorem 1: Under this nonpersistent user model with any success probability function $\theta(b)$ that satisfies (1)-(3) we have

a) The CTMC is reversible and the unique steady state distribution is, for all $(x_1, \ldots, x_k) \in \mathcal{S}$:

$$p(x_1, ..., x_k) = A \cdot \left(\prod_{r=0}^{-1+x_1+\ldots+x_k} \theta(r) \right) \prod_{i=1}^{k} \frac{\rho_i^{x_i}}{x_i!} \tag{7}$$

where A is the positive constant that makes the probabilities sum to 1, and we use the convention that $\prod_{r=0}^{-1} \theta(r) = 1$.

b) The steady state probability that b channels are in use is

$$P\left[\sum_{i=1}^{k} X_i = b\right] = A \cdot \left(\prod_{r=0}^{b-1} \theta(r) \right) \frac{\rho^b}{b!} \quad \forall b \in \{0, 1, \ldots, m\}$$

where $(X_1, ..., X_k)$ represents a random state vector with distribution equal to the steady state distribution.

c) The constant A is equal to

$$A = p(0, 0, ..., 0) = \frac{1}{\sum_{b=0}^{m} \left(\prod_{r=0}^{b-1} \theta(r) \right) \frac{\rho^b}{b!}} \tag{8}$$

Proof: Define the mass function $p(x)$ according to (7). It suffices to show that this $p(x)$ mass function satisfies (6). Since there are only two types of possible transitions, it suffices to show that

$$p(x)\lambda_j \theta \left(\sum_{i=1}^{k} x_i \right) = p(x + e_j)(x_j + 1)\mu_j \quad \forall x, x + e_j \in \mathcal{S}$$

Substituting the guess $p(x)$ defined in (7) verifies the above equations hold. The guess is correct. This proves part (a).

To prove (b), we have

$$\begin{aligned} P&\left[\sum_{i=1}^{k} X_i = b\right] \\ &= \sum_{x\in\mathcal{S}:(x_1+\ldots+x_k)=b} p(x) \\ &= \sum_{x\in\mathcal{S}:(x_1+\ldots+x_k)=b} A \cdot \left(\prod_{r=0}^{b-1} \theta(r) \right) \prod_{i=1}^{k} \frac{\rho_i^{x_i}}{x_i!} \\ &= A \cdot \left(\prod_{r=0}^{b-1} \theta(r) \right) \underbrace{\sum_{x\in\mathcal{S}:(x_1+\ldots+x_k)=b} \left(\prod_{i=1}^{k} \frac{\rho_i^{x_i}}{x_i!} \right)}_{(\rho_1+\ldots+\rho_k)^b/b!} \end{aligned}$$

where we have used the multinomial expansion:

$$(\rho_1 + \ldots + \rho_k)^b = \sum_{x\in\mathcal{S}:(x_1+\ldots+x_k)=b} \left(\frac{b!}{x_1!x_2!\ldots x_k!} \right) \prod_{i=1}^{k} \rho_i^{x_i}$$

This proves (b). Part (c) immediately follows from part (b). ■

The success probability of each newly arriving job depends on the current state of the system and not on the class of that job. Since jobs of each class arrive as Poisson arrivals, and Poisson arrivals see time averages ("PASTA," see, for example, [24]), it follows that jobs of all classes $i \in \{1, ..., k\}$ have the same *long term success probability* for finding an available channel. Define ϕ as this long term success probability. Specifically, if $(x_1, ..., x_k)$ represents a random vector with distribution given by the steady state distribution $p(x)$ in Theorem 1, then ϕ is defined

$$\phi = P[\text{success}] = \sum_{x\in\mathcal{S}} P[\text{success}|(x_1, ..., x_k) = x]p(x)$$

With this definition of the success probability ϕ, the long term rate of accepted jobs of type i is $\lambda_i\phi$ jobs/time, and the long term rate of dropped jobs of type i is $\lambda_i(1-\phi)$ jobs/time. Remarkably, the value of ϕ depends only on ρ, not on the individual ρ_i values, as shown in the following corollary.

Corollary 1: For any function $\theta(b)$ that satisfies (1)-(3), the long term success probability ϕ is given by

$$\phi = A \cdot \sum_{b=0}^{m} \left(\prod_{r=0}^{b} \theta(r)\right) \frac{\rho^b}{b!} \tag{9}$$

where A is the constant defined in (8).

Proof: The long term success probability is given by

$$\begin{aligned}\phi &= P[\text{success}] \\ &= \sum_{b=0}^{m} \underbrace{P\left[\text{success}|\textstyle\sum_{i=1}^{k} X_i = b\right]}_{\theta(b)} P\left[\sum_{i=1}^{k} X_i = b\right]\end{aligned}$$

and the result follows by substituting $P\left[\sum_{i=1}^{k} X_i = b\right]$ from part (b) of Theorem 1. ■

Corollary 1 shows that the success probability ϕ depends only on the conditional success probabilities $\theta(b)$ and on the loading parameter $\rho = \sum_{i=1}^{k} \lambda_i/\mu_i$. This means that we can understand the success probability through the single parameter ρ, regardless of the number of classes k of nonpersistent users and regardless of the specific λ_i and μ_i parameters for each class $i \in \{1, \ldots, k\}$. Notice that, by Little's theorem, $\rho = \sum_{i=1}^{k} \lambda_i/\mu_i$ is equal to the steady state average number of actively transmitting users there would be in a virtual system with *infinite resources*: The virtual system has an infinite number of servers, each new file of the virtual system receives its own server with probability 1, and no files are dropped.

D. Plots for example cases

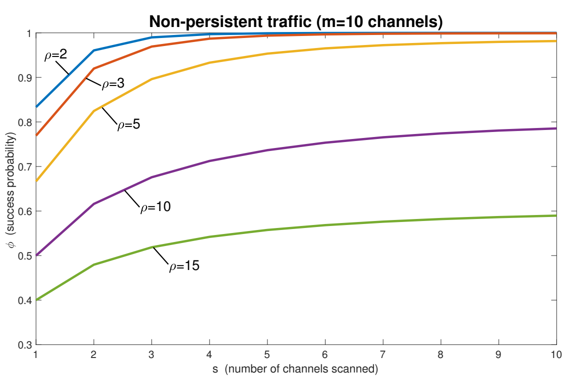

Fig. 3. Success probability for nonpersistent traffic with $m = 10$ channels.

Fig. 3 plots the success probability ϕ versus s (the number of channels that each user scans) for the case of $m = 10$ channels and using the $\theta(b)$ probabilities given in (4). The values $\rho \in \{2, 3, 5, 10, 15\}$ are shown. The case $\rho = 10$ is when the average number of active users in a virtual system with infinite resources is equal to 10, the number of channels in the actual system. This can be viewed as a threshold case: When ρ exceeds m (as plotted for the case $\rho = 15$ in Fig. 3) then success probability is necessarily strictly less than 1 even when the number of channels sensed is equal to m. On the other hand, by choosing $s = 2$ we obtain a success probability above 0.8 when $\rho \in \{2, 3, 5\}$.

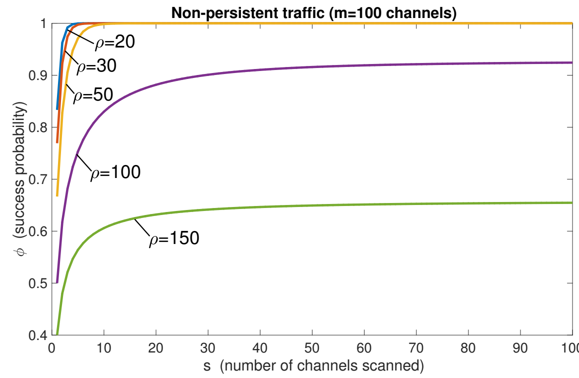

Fig. 4. Success probability for nonpersistent traffic with $m = 100$ channels.

Better performance is obtained when the number of channels is increased while the ρ values are increased by the same factor: Fig. 4 shows performance for the case $m = 100$ channels, with corresponding ρ values that maintain the same ratio of ρ/m as in the first figure. It can be seen that success probability increases to near 1 when the loading is small ($\rho/m \leq 1/2$). In all of the plots of Figs. 3-4 it can be seen that success probability is relatively flat for large values of s: A considerable amount of energy can be saved by just scanning a small subset of the total number of channels.

III. Persistent and nonpersistent users

Fix n as a positive integer and suppose that, in addition to the k classes of nonpersistent users, there are n individual persistent users with activity states $A_j(t) \in \{I, W, T\}$ and behavior parameters $\alpha_j, \beta_j, u_j, v_j$, as shown in Fig. 1. The k classes of nonpersistent users have parameters λ_i and μ_i for all $i \in \{1, \ldots, k\}$. The values of all parameters λ_i, μ_i, $\alpha_j, \beta_j, u_j, v_j$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$ are assumed to be positive.

Recall that $X_i(t)$ is the current number of nonpersistent users of type i transmitting, for $i \in \{1, \ldots, k\}$. The system state is $W(t) = (X_1(t), \ldots, X_k(t); A_1(t), \ldots, A_n(t))$. The total number of busy channels is

$$B(t) = \sum_{i=1}^{k} X_i(t) + \sum_{j=1}^{n} 1_{\{A_j(t)=T\}}$$

where $1_{\{A_j(t)=T\}}$ is an indicator function that is 1 if persistent user j is transmitting at time t, and 0 else. Let $\theta(b)$ be a success probability function defined for $b \in \{0, 1, \ldots, m\}$ that satisfies (1)-(3) (an example $\theta(b)$ function is in (4)). As before, if any user attempts access at a time t such that $B(t) = b$, its conditional success probability is $\theta(b)$. Notice from Fig. 1 that the transition rates for the $W \to T$ transitions of each persistent user depend on the current value of $B(t)$.

A. Markov chain model

Let $\mathcal{S}$ be the state space of the system: This is the set of all vectors $w = (x; a)$, where $x = (x_1, \ldots, x_k)$ and $a = (a_1, \ldots, a_n)$, such that $x_i \in \{0, 1, 2, \ldots\}$ for all $i \in \{1, \ldots, k\}$, $a_j \in \{I, W, T\}$ for all $j \in \{1, \ldots, n\}$, and

$$\sum_{i=1}^{k} x_i + \sum_{j=1}^{n} 1_{\{a_j = T\}} \leq m$$

For simplicity of notation, for each $w \in \mathcal{S}$ define busy(w) as the number of busy channels associated with state w:

$$\text{busy}(w) = \sum_{i=1}^{k} x_i + \sum_{j=1}^{n} 1_{\{a_j = T\}}$$

To completely describe the transition rates of this CTMC, let $w = (x; a)$ and $z = (y; b)$ be two distinct states in $\mathcal{S}$. There are three types of transitions that can occur between states w and z:

- Non-persistent user $i \in \{1, \ldots, k\}$ ($x_i \leftrightarrow x_i + 1$): Recall that $e_i = (0, ..., 0, 1, 0, ..., 0)$ is a vector of size k with a 1 in component i and zeros in all other components.
  - Transitions $(x; a) \to (x + e_i; a)$ have rate
  $$q_{w,z} = \lambda_i \theta(\text{busy}(w))$$
  - Transitions $(x + e_i; a) \to (x; a)$ have rate
  $$q_{z,w} = (x_i + 1)\mu_i$$
- Persistent user $j \in \{1, \ldots, n\}$ ($I \leftrightarrow W$, see Fig. 1):
  - Transitions
  $$\begin{aligned}&(x, a_1, \ldots, a_{j-1}, I, a_{j+1}, \ldots, a_n)\\ &\to (x, a_1, \ldots, a_{j-1}, W, a_{j+1}, \ldots, a_n)\end{aligned}$$
  have rate $q_{w,z} = \alpha_j$.
  - Transitions
  $$\begin{aligned}&(x, a_1, \ldots, a_{j-1}, W, a_{j+1}, \ldots, a_n)\\ &\to (x, a_1, \ldots, a_{j-1}, I, a_{j+1}, \ldots, a_n)\end{aligned}$$
  have rate $q_{z,w} = \beta_j$.
- Persistent user $j \in \{1, \ldots, n\}$ ($W \leftrightarrow T$, see Fig. 1):
  - Transitions
  $$\begin{aligned}&(x, a_1, \ldots, a_{j-1}, W, a_{j+1}, \ldots, a_n)\\ &\to (x, a_1, \ldots, a_{j-1}, T, a_{j+1}, \ldots, a_n)\end{aligned}$$
  have rate $q_{w,z} = u_j \theta(\text{busy}(w))$.
  - Transitions
  $$\begin{aligned}&(x_0, a_1, \ldots, a_{j-1}, T, a_{j+1}, \ldots, a_n)\\ &\to (x_0, a_1, \ldots, a_{j-1}, W, a_{j+1}, \ldots, a_n)\end{aligned}$$
  have rate $q_{z,w} = v_j$.

It is not difficult to show that the CTMC is irreducible. Indeed, every state can reach the state $(0, I, I, I, ..., I)$ from a sequence of transitions that includes no new arrivals, has each transmitting user finish, and has all persistent users eventually move to the Idle state. Likewise, the state $(0, I, I, I, ..., I)$ can reach every state in $\mathcal{S}$.

B. Steady state probabilities

Motivated by the "birth-death-like" structure of the persistent user dynamics shown in Fig. 1 and by the structure of the steady state probabilities for the nonpersistent user case, we make the following guess about steady state: With $w = (x_1, \ldots, x_k; a_1, \ldots, a_n)$ we guess that for all $w \in \mathcal{S}$:

$$\begin{aligned}p(w) = B \cdot &\left(\prod_{r=0}^{\text{busy}(w)-1} \theta(r)\right)\left(\prod_{i=1}^{k} \frac{\rho_i^{x_i}}{x_i!}\right)\\ &\times \prod_{j=1}^{n} \left(\frac{\alpha_j}{\beta_j}\right)^{1_{\{a_j=W\}}} \left(\frac{\alpha_j u_j}{\beta_j v_j}\right)^{1_{\{a_j=T\}}}\end{aligned} \tag{10}$$

where $\rho_i = \lambda_i / \mu_i$ for all $i \in \{1, \ldots, k\}$ and B is a constant that makes all probabilities sum to 1.

Theorem 2: The CTMC for this system with persistent and nonpersistent users is reversible and the steady state distribution is given by (10).

Proof: It suffices to show that $p(w)$ defined by (10) satisfies the detailed balance equations. The proof is omitted for brevity (see [28]). ∎

The steady state probabilities in the above theorem can be simplified by aggregating all nonpersistent users. Consider a state $w = (x_1, \ldots, x_k; a_1, \ldots, a_n) \in \mathcal{S}$. Define $x = \sum_{i=1}^{k} x_i$ as the number of nonpersistent users associated with this state. Define $q(x; a_1, \ldots, a_n)$ as the steady state probability that the total number of nonpersistent users is x and the state of the persistent users is $(a_1, \ldots, a_n)$. Define $a = (a_1, \ldots, a_n)$ and define

$$\text{busy}_p(a) = \textstyle\sum_{j=1}^{n} 1_{\{a_j=T\}}$$

where the p subscript emphasizes that busy$_p(a)$ counts the number of busy *persistent* users from the vector $a = (a_1, \ldots, a_n)$. In particular, the total number of busy channels for a vector (x, a) is $x + \text{busy}_p(a)$. A vector (x, a) is said to be a *legitimate vector* if $x + \text{busy}_p(a) \leq m$. The next result allows the aggregatated steady state to be written purely in terms of ρ.

Corollary 2: For this system with persistent and nonpersistent users we have for all legitimate vectors $(x, a_1, \ldots, a_n)$:

$$\begin{aligned}&q(x; a_1, \ldots, a_n)\\ &= B \cdot \left(\prod_{r=0}^{x+\text{busy}_p(a)-1} \theta(r)\right)\\ &\quad \times \prod_{j=1}^{n} \left(\frac{\alpha_j}{\beta_j}\right)^{1_{\{a_j=W\}}} \left(\frac{\alpha_j u_j}{\beta_j v_j}\right)^{1_{\{a_j=T\}}} \frac{\rho^x}{x!}\end{aligned} \tag{11}$$

where B is the same constant used in Theorem 2, $\rho = \sum_{i=1}^{k} \rho_i$, and $\rho_i = \lambda_i / \mu_i$ for $i \in \{1, \ldots, k\}$.

Proof: Omitted for brevity (see [28]). ∎

C. Solution complexity

The formulas (10) and (11) establish steady state probabilities for a very large number of system states. The number of states grows exponentially in the problem size. For example, just considering the 3 possibilities I, W, T for each persistent user, we find the number of states is at least $3^{\min[n,m]}$. If $\min[n,m] \geq 180$ then $3^{\min[n,m]} \geq 10^{85}$, meaning that the number of states is larger than the current estimate for the number of atoms in the universe. Thus, it is not immediately clear how to compute the constant B, and how to use the formulas (10) and (11) to calculate things such as the marginal fraction of time that persistent user 1 is busy, the throughput and success probability of persistent user 1, and the throughput and success probabilities of the different classes of nonpersistent users. For some problems that involve reversible networks, such as the admission control problems in [24], it can be shown that even calculating the proportionality constant B to within a reasonable approximation is NP-hard [16] (see also [17] for factor graph approximation methods). Fortunately, our problem has enough structure to allow efficient (polynomial time) computation of all of these things via a discrete Fourier transform. That is developed next.

D. Calculating B

Define

$$g = \min[m, n]$$

We can sum the probabilities in (11) by grouping states into those that have b persistent users that are busy, for $b \in \{0, 1, \ldots, g\}$, and x nonpersistent users:

$$\begin{aligned} 1 &= \sum_{b=0}^{g}\sum_{x=0}^{m-b}\sum_{a:\text{busy}_p(a)=b} q(x, a_1, \ldots, a_n) \\ &= B\sum_{b=0}^{g}\sum_{x=0}^{m-b}\frac{\rho^x}{x!}\left(\prod_{r=0}^{x+b-1}\theta(r)\right) \\ &\quad\times \sum_{a:\text{busy}_p(a)=b}\prod_{j=1}^{n}\left(\frac{\alpha_j}{\beta_j}\right)^{1\{a_j=W\}}\left(\frac{\alpha_j u_j}{\beta_j v_j}\right)^{1\{a_j=T\}} \\ &= B\sum_{b=0}^{g}\sum_{x=0}^{m-b}\frac{\rho^x}{x!}\left(\prod_{r=0}^{x+b-1}\theta(r)\right)c_b \end{aligned} \tag{12}$$

where we define c_b for all $b \in \{0, 1, \ldots, n\}$

$$c_b = \sum_{a:\text{busy}_p(a)=b}\prod_{j=1}^{n}\left(\frac{\alpha_j}{\beta_j}\right)^{1\{a_j=W\}}\left(\frac{\alpha_j u_j}{\beta_j v_j}\right)^{1\{a_j=T\}} \tag{13}$$

Notice that these c_b values are defined for all $b \in \{0, 1, \ldots, n\}$, even if the number of persistent users n is larger than the number of channels m (so that only $c_0, \ldots, c_m$ are used in (12)).

It is difficult to obtain the value of c_b by a direct summation in (13) because there are so many terms. However, we can construct a related polynomial function $f(z)$ defined for all complex numbers $z \in \mathbb{C}$:

$$f(z) = \prod_{j=1}^{n}\left[1 + \left(\frac{\alpha_j}{\beta_j}\right) + z\left(\frac{\alpha_j u_j}{\beta_j v_j}\right)\right]$$

For any given $z \in \mathbb{C}$, the value $f(z)$ can be easily computed as a product of n (complex-valued) terms. We make the crucial observation:

$$f(z) = \sum_{b=0}^{n} c_b z^b$$

This motivates a discrete Fourier transform approach: Define $i = \sqrt{-1}$ and define

$$C_t = f(e^{\frac{-2\pi i t}{n+1}}) = \sum_{b=0}^{n} c_b e^{\frac{-2\pi i b t}{n+1}} \quad \forall t \in \{0, 1, \ldots, n\}$$

The sequence $\{C_t\}_{t=0}^{n}$ is the discrete Fourier transform of $\{c_b\}_{b=0}^{n}$. The inverse transform gives

$$c_b = \frac{1}{n+1}\sum_{t=0}^{n} C_t e^{\frac{2\pi i t b}{n+1}} \quad \forall b \in \{0, 1, \ldots, n\}$$

Of course, these values of c_b only need to be computed for $b \in \{0, 1, \ldots, g\}$ for use in (12). These findings are summarized in the following lemma.

Lemma 1: (Calculating B) The value B in Theorem 2 and Corollary 2 is

$$B = \frac{1}{\sum_{b=0}^{g}\sum_{x=0}^{m-b}\frac{\rho^x}{x!}\left(\prod_{r=0}^{x+b-1}\theta(r)\right)c_b} \tag{14}$$

where $g = \min[n, m]$ and c_b is defined

$$c_b = \frac{1}{n+1}\sum_{t=0}^{n} C_t e^{\frac{2\pi i t b}{n+1}} \quad \forall b \in \{0, 1, \ldots, g\} \tag{15}$$

with $i = \sqrt{-1}$ and with C_t given by

$$C_t = \prod_{j=1}^{n}\left[1 + \left(\frac{\alpha_j}{\beta_j}\right) + e^{\frac{-2\pi i t}{n+1}}\left(\frac{\alpha_j u_j}{\beta_j v_j}\right)\right] \quad \forall t \in \{0, 1, \ldots, n\}$$

Proof: The proof is contained in the development immediately preceding the lemma. ■

E. Performance for the individual persistent users

Fix $j \in \{1, \ldots, n\}$. Define $P[I_j]$, $P[W_j]$ and $P[T_j]$ as the steady state probability that persistent user j is idle, waiting, or transmitting, respectively (see Fig. 1). Define performance variables γ_j and ϕ_j as follows:

- γ_j is the *throughput* of persistent user j. This is the rate at which this user successfully accesses a channel of the multi-access system. Because all files that successfully access a channel are eventually served, γ_j is also the rate of file service for persistent user j and so

$$\gamma_j = P[T_j]v_j \tag{16}$$

- ϕ_j is the *success ratio* of persistent user j. This is the rate of access successes divided by the rate of access attempts:

$$\phi_j = \frac{\gamma_j}{P[W_j]u_j} \tag{17}$$

The next two lemmas show that: (i) These values can be obtained in terms of $P[I_j]$; (ii) The probability $P[I_j]$ can be computed via the discrete Fourier transform.

Lemma 2: For persistent user $j \in \{1, \ldots, n\}$ we have

$$\begin{aligned} P[W_j] &= P[I_j](\alpha_j/\beta_j) \\ P[T_j] &= 1 - P[I_j](1 + (\alpha_j/\beta_j)) \\ \gamma_j &= v_j - v_j P[I_j](1 + (\alpha_j/\beta_j)) \\ \phi_j &= \frac{v_j\beta_j - v_j P[I_j](\beta_j + \alpha_j)}{u_j\alpha_j P[I_j]} \end{aligned}$$

Proof: We use an argument similar to *cut set equations* for CTMCs (in this case we use cuts on the incompletely described CTMC of Fig. 1): Consider the 3-state picture of Fig. 1 associated with persistent user j. The total number of transitions $I \rightarrow W$ is always within 1 of the number of transitions $W \rightarrow I$. Hence, the long term time average rate of transitions $I \rightarrow W$ (in units of transitions/time) is the same as the long term rate for transitions $W \rightarrow I$:

$$P[I_j]\alpha_j = P[W_j]\beta_j \implies P[W_j] = P[I_j](\alpha_j/\beta_j)$$

On the other hand we know $P[I_j] + P[W_j] + P[T_j] = 1$ and so

$$P[T_j] = 1 - P[I_j](1 + (\alpha_j/\beta_j))$$

The resulting values of γ_j and ϕ_j are obtained by (16) and (17). ■

Lemma 3: For each persistent user $j \in \{1, \ldots, n\}$, the value of $P[I_j]$ is

$$P[I_j] = B \sum_{b=0}^{\min[n-1,m]} \sum_{x=0}^{m-b} \left(\prod_{r=0}^{x+b-1} \theta(r)\right) \frac{\rho^x}{x!} c_{j,b} \quad (18)$$

where B is given in (14), $\rho = \sum_{i=1}^{k}(\lambda_i/\mu_i)$, and $c_{j,b}$ is defined by

$$c_{j,b} = \frac{1}{n}\sum_{t=0}^{n-1} \prod_{l\in\{1,\ldots,n\}\setminus j} \left[1 + \left(\frac{\alpha_l}{\beta_l}\right) + e^{\frac{-2\pi i t}{n}} \left(\frac{\alpha_l u_l}{\beta_l v_l}\right)\right] e^{\frac{2\pi i t b}{n}}$$
$$\forall b \in \{0, 1, \ldots, \min[n-1, m]\} \quad (19)$$

Proof: Omitted for brevity (see [28]). ■

F. Performance for nonpersistent users

Recall that nonpersistent users arrive according to independent Poisson arrival processes. Since Poisson arrivals see time averages (PASTA), it holds that the fraction of time that nonpersistent users of class $i \in \{1, \ldots, k\}$ see a system with y busy channels is the same for all classes i and is equal to the long term fraction of time that there are y busy channels. Hence, all nonpersistent users see the same access success probability, call it ϕ_0, and we can show (details omitted for brevity, see [28]):

$$\phi_0 = B \sum_{y=0}^{m} \sum_{b=0}^{\min[y,n]} \frac{\rho^{y-b}}{(y-b)!} \left(\prod_{r=0}^{y}\theta(r)\right) c_b$$

IV. Validation on test cases

This section validates the results of the previous section (obtained by the discrete Fourier transform) by considering simple example cases and comparing to simulated performance.

A. Three identical persistent users

Consider the following test case with $m = 5$ channels and where each user scans a subset $s = 2$ of these channels. There is one class of nonpersistent traffic with $\lambda_1 = 1$ and $\mu_1 = 2$, so that $\rho = 1/2$. There are three persistent users with identical parameters $\alpha_j = \alpha$, $\beta_j = \beta$, $u_j = u$, $v_j = v$ for all $j \in \{1, 2, 3\}$ with

$$\alpha = 1; \beta = 1; u = 5; v = 10$$

We compare the exact success probabilities $\phi_0, \phi_1, \phi_2, \phi_3$ and the persistent user state probabilities $P[I_j]$, $P[W_j]$, $P[T_j]$ obtained by the formulas in the previous section with simulation values. The exact values were calculated using complex number multiplication in MATLAB (the imaginary parts of all real-valued quantities were indeed found to be zero in the MATLAB computation). The simulation was conducted in MATLAB over a period of 10^7 transitions of the CTMC. Access attempts that fail were also counted as transitions, even though these transitions did not change the state of the CTMC.

Tables I-II report the results. The data from these tables shows a good agreement between theory and simulation.

Type	Success prob	Simulated	Exact
Non-persistent	ϕ_0	0.9530	0.9527
Persistent	ϕ_1	0.9676	0.9674
Persistent	ϕ_2	0.9675	0.9674
Persistent	ϕ_3	0.9670	0.9674

TABLE I
A COMPARISON OF SIMULATED AND EXACT VALUES FOR SUCCESS PROBABILITY.

Persistent Users	$P[I_j]$	$P[W_j]$	$P[T_j]$
User 1 Simulation	0.4028	0.4024	0.1948
User 2 Simulation	0.4021	0.4030	0.1949
User 3 Simulation	0.4035	0.4019	0.1946
Exact	0.4026	0.4026	0.1947

TABLE II
A COMPARISON OF SIMULATED AND EXACT VALUES FOR STATE PROBABILITIES.

B. Two classes of persistent users (three in each class)

This test case considers $m = 10$ channels and $s = 2$. This test case considers two classes of persistent users (as defined by the $\alpha_j, \beta_j, u_j, v_j$ parameters) with three users in each class:

- Persistent class A: $\alpha = 1; \beta = 1; u = 5; v = 10$
- Persistent class B: $\alpha = 1; \beta = 1; u = 5; v = 1$

In particular, class A persistent users have the same parameters as the previous subsection, while class B persistent users have file that take 10 times longer to serve. There is a single nonpersistent class with parameters $\lambda_1 = 1, \mu_1 = 1$ (so that $\rho = 1$).

As before, we compare the exact theoretical values from the previous section with simulated values obtained over a simulation with 10^7 transitions. The results are shown in Tables III-V. Again there is good agreement between theory and simulation. Notice that class B persistent users spend much more time in the transmitting state. These users also have a slightly higher access success probability. Intuitively, this is because the average time that a class B user spends transmitting a file over a channel is 10 times longer than class A users and nonpersistent users. Thus, when a class B user wants to access a channel, this particular class B user is certainly *not currently occupying a channel*, which is one less channel-hogger to worry about.

C. Scaling the number of channels

We now consider scaling the network size by a scale parameter $k \in \{1, 2, \ldots, 10\}$. The number of channels scanned is

Type	Success prob	Simulated	Exact
Non-persistent	ϕ_0	0.8819	0.8822
Persistent A	ϕ_1	0.8940	0.8937
Persistent A	ϕ_2	0.8938	0.8937
Persistent A	ϕ_3	0.8939	0.8937
Persistent B	ϕ_4	0.9213	0.9209
Persistent B	ϕ_5	0.9206	0.9209
Persistent B	ϕ_6	0.9207	0.9209

TABLE III
TWO CLASSES OF PERSISTENT USERS: A COMPARISON OF SIMULATED AND EXACT VALUES FOR SUCCESS PROBABILITY.

Persistent Class A	$P[I_j]$	$P[W_j]$	$P[T_j]$
User 1 Simulation	0.4086	0.4090	0.1823
User 2 Simulation	0.4094	0.4084	0.1822
User 3 Simulation	0.4093	0.4082	0.1825
Exact	0.4087	0.4087	0.1826

TABLE IV
TWO CLASSES OF PERSISTENT USERS: A COMPARISON OF STATE PROBABILITIES FOR CLASS A PERSISTENT USERS.

fixed at $s = 2$. For each $k \in \{1, 2, \ldots, 10\}$ there are $m = 20k$ channels (so the number of channels increases from 20 to 200). There are three classes of users and the number of users in each class increases linearly with k:

- Non-persistent users: There is a single class of nonpersistent users with $\mu_1 = 1$ and $\lambda_1 = 3k$ (so $\rho = 3k$).
- Type A persistent users: There are $5k$ persistent users of Class A, defined by parameters:

$$[\alpha_A, \beta_A, u_A, v_A] = [0.5, 0.5, 5, 10]$$

- Type B persistent users: There are $5k$ persistent users of Class B, defined by parameters:

$$[\alpha_B, \beta_B, u_B, v_B] = [0.5, 0.5, 5, 1]$$

Notice that Type B persistent users and Non-persistent users have the same average file size of $1/\mu_1 = 1/v_B = 1$, while Type A persistent users have average file size that is 10 times smaller ($1/v_A = 1/10$). The data is plotted in Fig. 5. The solid curves of Fig. 5 represent exact values calculated by the formulas of the previous section, while the diamonds correspond to simulated values over a simulation of 10^7 transitions.

V. CONCLUSION

This paper considers a Markov chain model for wireless multi-channel multiple access with heterogenous users. When all users are nonpersistent and send at most one file, a simple expression for success probability was derived that depends only on m, s, and the system loading $\rho = \sum_{i=1}^{k} \lambda_i/\mu_i$, where k is the (arbitrarily large) number of user classes and λ_i, μ_i

Persistent Class B	$P[I_j]$	$P[W_j]$	$P[T_j]$
User 4 Simulation	0.1507	0.1509	0.6984
User 5 Simulation	0.1533	0.1512	0.6955
User 6 Simulation	0.1515	0.1510	0.6976
Exact	0.1514	0.1514	0.6972

TABLE V
TWO CLASSES OF PERSISTENT USERS: A COMPARISON OF STATE PROBABILITIES FOR CLASS B PERSISTENT USERS.

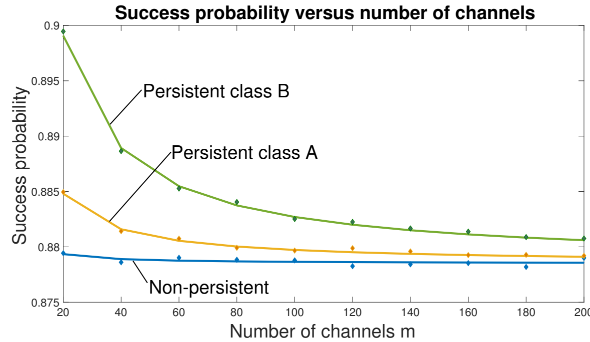

Fig. 5. Success probability versus network size. The solid curves are computed from the exact formula; the diamonds show corresponding simulated values for simulations over 10^7 transitions of the CTMC. The number of channels scanned is fixed at $s = 2$.

are the arrival rate and file size parameters of each class. The case with both persistent and nonpersistent users was also analyzed. Each persistent user has its own *activity parameters* and behaves according to a 3-state process with idle, waiting, and transmitting states. The exact steady state values were also derived in this setting. An efficient method for summing over the overwhelmingly large number of state probabilities to obtain the individual performance of each user was developed using a discrete Fourier transform.

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