Review of Basic Calculus

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Abstract

These notes review basic calculus, including the chain rule, product rule, and the fundamental theorems of calculus. The rules for integration by substitution and integration by parts are derived as consequences. Finally, worked examples that apply the substitution rule and the integration by parts rule are given. The functions $f(x)$ considered in these notes are assumed to be real valued functions defined over $x \in \mathbb{R}$, that is, $f : \mathbb{R} \rightarrow \mathbb{R}$.

I. Basic calculus results

A. *Differentiation*

Let $f(x), g(x)$ be differentiable functions and let $f'(x), g'(x)$ be the derivatives.

- Chain rule: If $h(x) = f(g(x))$ then $h'(x) = f'(g(x))g'(x)$.
- Product rule: If $h(x) = f(x)g(x)$ then $h'(x) = f'(x)g(x) + f(x)g'(x)$.

B. *Fundamental theorems of calculus*

- $\int_a^b f'(x)dx = f(b) - f(a)$. [Assuming $f'(x)$ is continuous at all points $x \in [a, b]$.]
- $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$. [Assuming $f(t)$ is continuous at the point x.]

II. Integration techniques

A. *Substitution rule for integration*

If $f(u)$, $g(x), g'(x)$ are continuous functions over their domains, and if a, b are given real numbers, then

$$\boxed{\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du}$$

In particular, the integral is zero in the special case when $g(a) = g(b)$.

Proof: (Application of chain rule for differentiation) Define $h(x) = F(g(x))$, where $F(u)$ is an antiderivative of $f(u)$ (meaning that $F'(u) = f(u)$). By the *chain rule* we get:

$$h'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$$

Integrating both sides gives:

$$\int_a^b h'(x)dx = \int_a^b f(g(x))g'(x)dx$$

On the other hand, we have:

$$\int_a^b h'(x) \overset{(a)}{=} h(b) - h(a) \overset{(b)}{=} F(g(b)) - F(g(a)) \overset{(c)}{=} \int_{g(a)}^{g(b)} F'(u)du = \int_{g(a)}^{g(b)} f(u)du$$

where (a) holds by the fundamental theorem of calculus applied to the continuous function $h'(x)$, (b) holds by definition of h, and (c) holds by the fundamental theorem of calculus applied to the continuous function $f(u)$. □

B. *Integration by parts*

If $f(x), f'(x), g(x), g'(x)$ are continuous functions and a, b are real numbers, then

$$\boxed{\int_a^b f(x)g'(x)dx = f(x)g(x)\,|_a^b - \int_a^b f'(x)g(x)dx}$$

where $f(x)g(x)\,|_a^b = f(b)g(b) - f(a)g(a)$.

Proof: (Application of product rule for differentiation) Define $h(x) = f(x)g(x)$. The *product rule* gives:

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

Integrating both sides gives:

$$\int_a^b h'(x)dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx$$

On the other hand,

$$\int_a^b h'(x)dx \overset{(a)}{=} h(b) - h(a) \overset{(b)}{=} f(b)g(b) - f(a)g(a)$$

where (a) holds by the fundamental theorem of calculus applied to the continuous function $h'(x)$, and (b) holds by definition of h. $\square$

C. *Examples of integration via the substitution rule*

Consider the equation for the substitution rule:

$$\int_a^b f(\underbrace{g(x)}_{u})\underbrace{g'(x)dx}_{du} = \int_{g(a)}^{g(b)} f(u)du$$

The substitution rule can be implemented by performing the change of variables

$$u = g(x)$$
$$du = g'(x)dx$$

and then remembering to appropriately change the integration limits.

1) Solving $\int_a^b xe^{x^2}dx$: Define $u = x^2$, $du = 2xdx$. Then:

$$\int_a^b xe^{x^2}dx = \int_{a^2}^{b^2} \frac{1}{2}e^u du = \frac{1}{2}e^u|_{a^2}^{b^2} = \frac{1}{2}(e^{b^2} - e^{a^2})$$

2) Solving $\int_1^3 e^x cos(e^x)dx$: Define $u = e^x, du = e^x dx$. Then:

$$\int_1^3 e^x cos(e^x)dx = \int_e^{e^3} \cos(u)du = \sin(e^3) - \sin(e)$$

D. *Examples of integration by parts*

Consider the integration by parts equation:

$$\int_a^b \underbrace{f(x)}_{u}\underbrace{g'(x)dx}_{dv} = \underbrace{f(x)}_{u}\underbrace{g(x)}_{v}\,|_a^b - \int_a^b \underbrace{g(x)}_{v}\underbrace{f'(x)dx}_{du}$$

Integration by parts can be implemented by the change of variables:

$$u = f(x) \quad , \quad dv = g'(x)dx$$
$$du = f'(x)dx \quad , \quad v = g(x)$$

1) Solving $\int_1^5 x\cos(x)dx$: Define $u = x, dv = \cos(x)dx$. Then:

$$\begin{aligned} u &= x \quad , \quad dv = \cos(x)dx \\ du &= dx \quad , \quad v = \sin(x) \end{aligned}$$

So:

$$\begin{aligned} \int_1^5 x\cos(x)dx &= x\sin(x)|_1^5 - \int_1^5 \sin(x)dx \\ &= x\sin(x)|_1^5 + \cos(x)|_1^5 \\ &= 5\sin(5) + \cos(5) - \sin(1) - \cos(1) \end{aligned}$$

2) Solving $\int_0^\infty xe^{-x}dx$: Define $u = x, dv = e^{-x}dx$. Then:

$$\begin{aligned} u &= x \quad , \quad dv = e^{-x}dx \\ du &= dx \quad , \quad v = -e^{-x} \end{aligned}$$

So:

$$\int_0^\infty xe^{-x}dx = -xe^{-x}|_0^\infty - \int_0^\infty (-e^{-x})dx = 0 + \int_0^\infty e^{-x}dx = -e^{-x}\,|_0^\infty = 1$$