CSCI 104
Skip Lists

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Sources / Reading

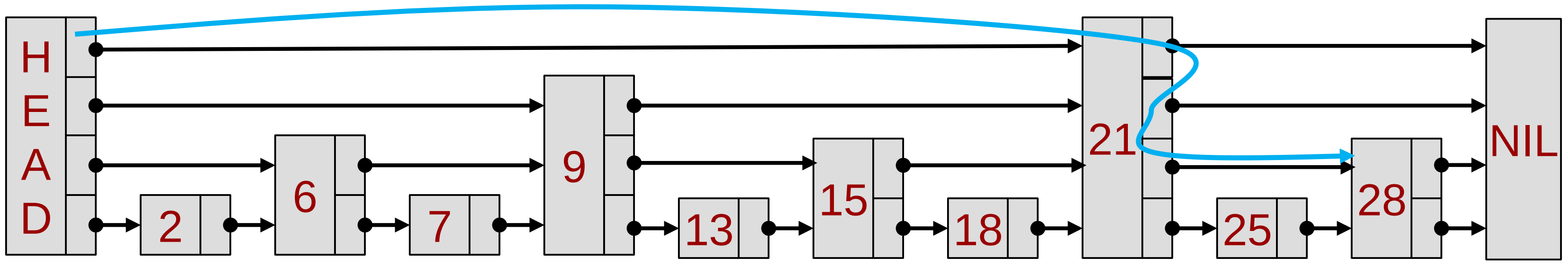
- Material for these slides was derived from the following sources
  - http://courses.cs.vt.edu/cs2604/spring02/Projects/1/Pugh.Skiplists.pdf
  - http://www.cs.umd.edu/~meesh/420/Notes/MountNotes/lecture11-skiplist.pdf

Skip List Intro

- Another map/set implementation (storing keys or key/value pairs)
  - Insert, Remove, Find
- Remember the story of Goldilocks and the Three Bears
  - **Father's porridge was too hot**
  - **Mother's porridge was too cold**
  - **Baby Bear's porridge was just right**
- Compare Set/Map implementations
  - BST's were easy but could degenerate to O(n) operations with an adversarial sequence of keys (too hot?)
  - Balanced BSTs guarantee O(log(n)) operations but are more complex to implement and may require additional memory overhead (too cold?)
  - Skip lists are fairly simple to implement, fairly memory efficient, and offer "expected" O(log(n)) operations (just right?)
    - Skip lists are a probabilistic data structure so we expect O(log(n))
    - Expectation of log(n) does not depend on keys but only random # generator

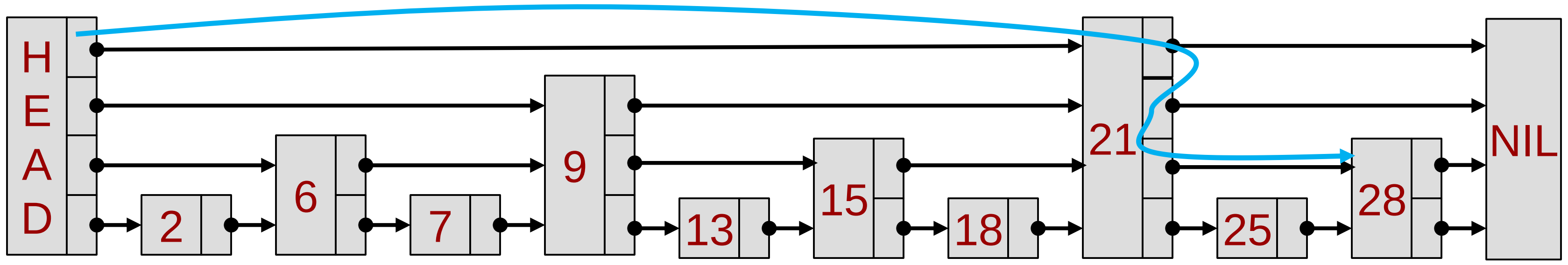
Skip List Visual

- Think of a skip list like a sorted linked list with shortcuts (wormholes?)
- Given the skip list below with the links (arrows) below what would be the fastest way to find if 28 is in the list?



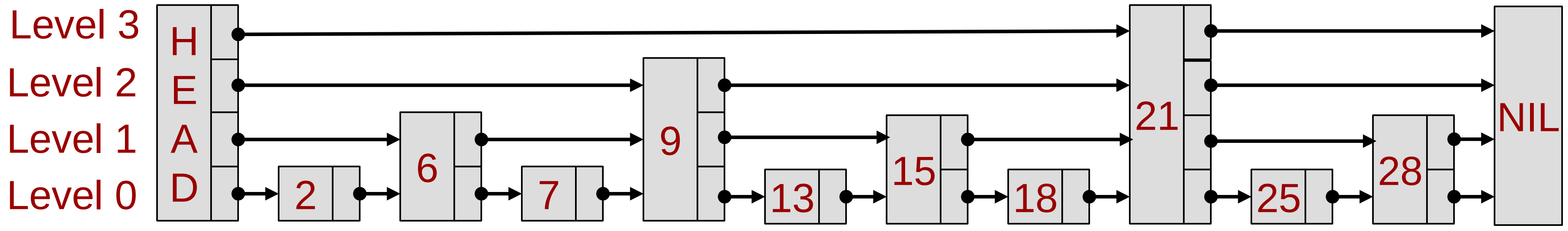
Skip List Visual

- Think of a skip list like a sorted linked list with shortcuts (wormholes?)
- Given the skip list below with the links (arrows) below what would be the fastest way to find if 28 is in the list?
  - Let p point to a node. Walk at level i while the desired search key is bigger than p->next->key, then descend to the level i-1 until you find the value or hit the NIL (end node)
  - NIL node is a special node whose stored key is BIGGER than any key we might expect (i.e. MAXKEY+1 / +infinity)



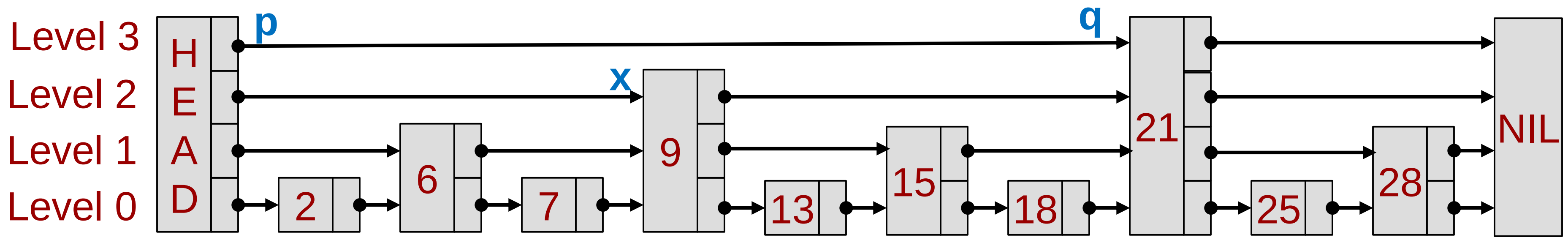
Perfect Skip List

- How did we form this special linked list?
  - We started with a normal linked list (level 0)
  - Then we took every other node in level 0 (2nd node from original list) and added them to level 1
  - Then we took every other node in level 1 (4th node from the original list) and raised it to level 2
  - Then we took every other node) in level 2 (8th node from the original list) and raised it to level 3
  - There will be O($\log_2(n)$) levels (We would have only 1 node at level $\log_2(n)$)



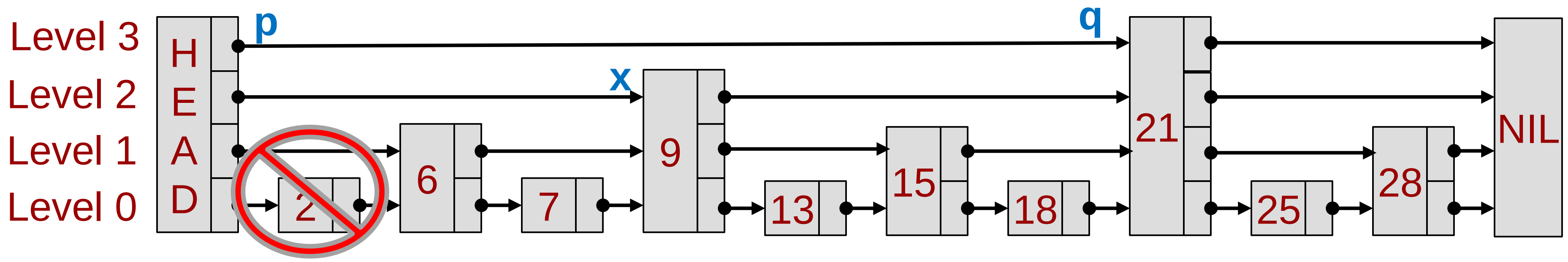
Search Time for Perfect Skip List

- How long would it take us to find an item or determine it is not present in the list
  - O(log(n))
- Proof
  - At each level we visit at most 2 nodes
    - At any node, x, in level i, you sit between two nodes (p,q) at level i+1 and you will need to visit at most one other node in level i before descending
  - There are O(log(n)) levels
  - So we visit at most O(2*log(n)) levels = O(log(n))



The Problem w/ Perfect Skip Lists

- Remember in a perfect skip list
  - Every 2nd node is raised to level 1
  - Every 4th node is raised to level 2
  - …
- What if I want to insert a new node or remove a node, how many nodes would need their levels adjusted to maintain the pattern described above?
  - In the worst case, all n-1 remaining nodes
  - Inserting/removing may require n-1 nodes to adjust



Quick Aside

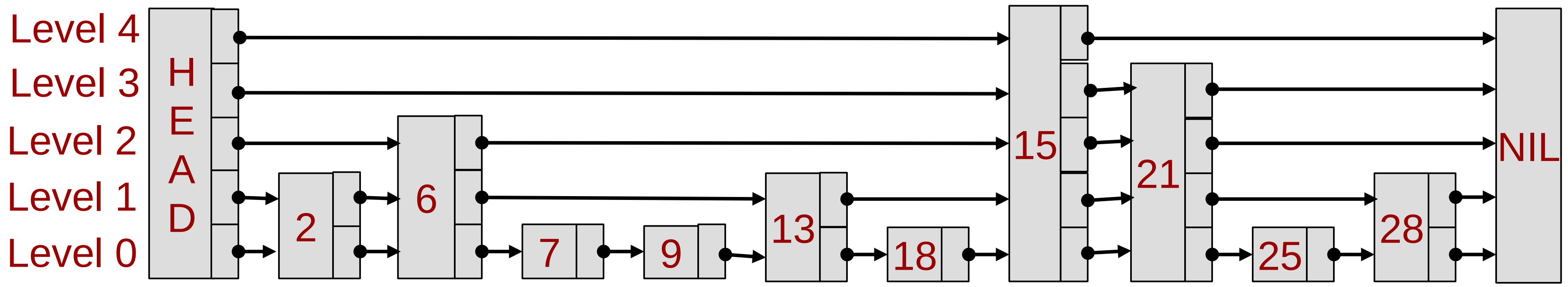
- Imagine a game where if you flip a coin and it comes up heads you get \$1 and get to play again. If you get tails you stop.
- What's the chance you win at least
  - \$1
  - \$2
  - \$3
- P(\$1)=1/2, P(\$2)=1/4, P(\$3)=1/8

Randomized Skip Lists

- Rather than strictly enforcing every other node of level i be promoted to level i+1 we simply use probability to give an "expectation" that every other node is promoted
- Whenever a node is inserted we will promote it to the next level with probability p (=1/2 for now)…we'll keep promoting it while we get heads
- What's the chance we promote to level 1, 2, 3?
- Given n insertions, how many would you expect to be promoted to:
  – Level 1 = n/2, Level 2 = n/4, Level 3 = n/8

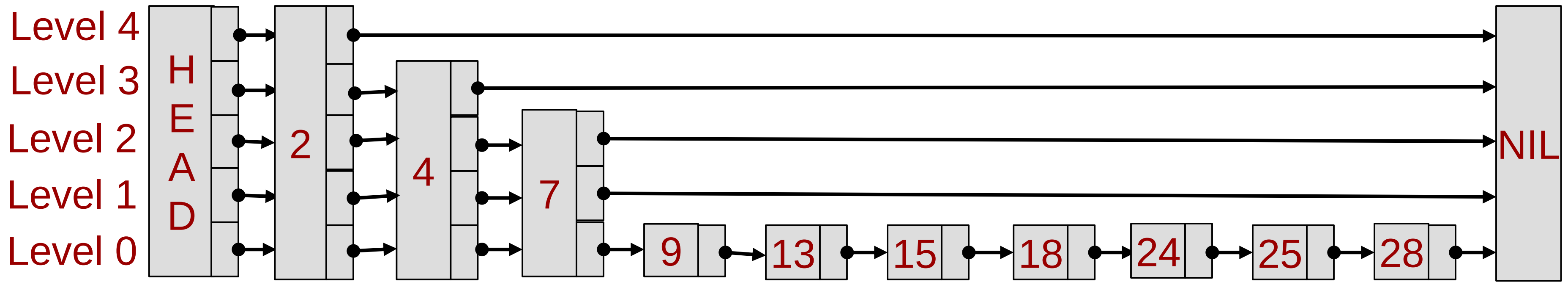
Randomized Skip List

- As nodes are inserted they are repeating trials of probability p (stopping when the first unsuccessful outcome occurs)
- This means we will not have an "every other" node promotion scheme, but the expected number of nodes at each level matches the non-randomized version
- Note: This scheme introduces the chance of some very high levels
  - We will usually cap the number of levels at some MAXIMUM value
  - However the expected number of levels is still $\log_2(n)$



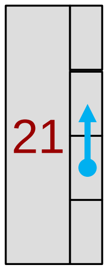
Worst Case

- What might a worst case skip list look like?
  - All the same height
  - Or just ascending or descending order of height
- These are all highly unlikely

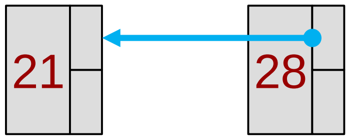


Analysis

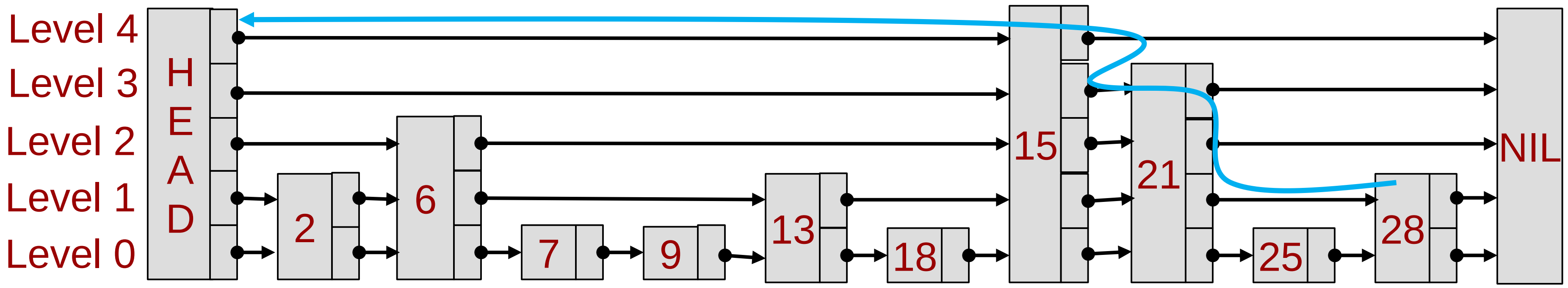
Option A:
If we can move up we do



Option B: No higher level, move left



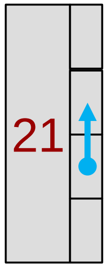
- To analyze the search time with this randomized approach let's start at the node and walk backwards to the head node counting our expected number of steps
  - Recall if we can move up a level we do, so that we take the "faster" path and only move left if we can't move up



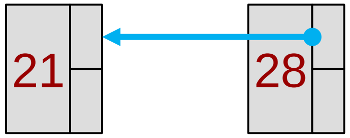
Analysis

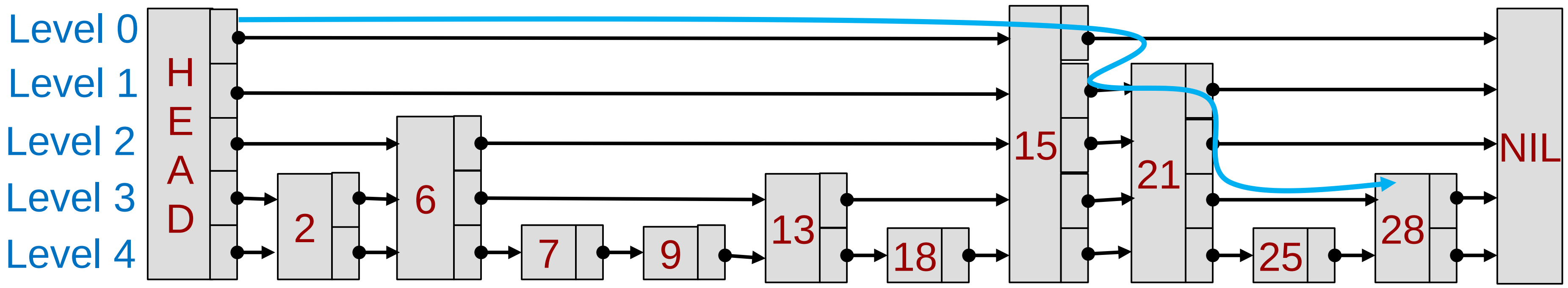
- Probability of Option A: p
  - Recall we added each level independently with probability p
- Probability of Option B: 1-p
- For this analysis let us define the top level at level 0 and the current level where we found our search node as level k (expected max k = $\log_2(n)$)

Option A: If we can move up we do



Option B: No higher level, move left

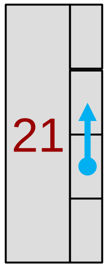




Analysis

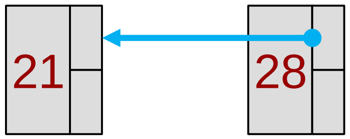
- Define a recurrence relationship of the cost of walking back from level k to level 0
- Base case: C(0) = O(1)
  - Only expect 1 node + head node at level 0
- Recursive case: C(k) = (1-p)(1+C(k)) + p(1+C(k-1))
  - 1+C(k) = Option B and its probability is (1-p)
  - 1+C(k-1) = Option A and its probability is p

Option A: If we can move up we do

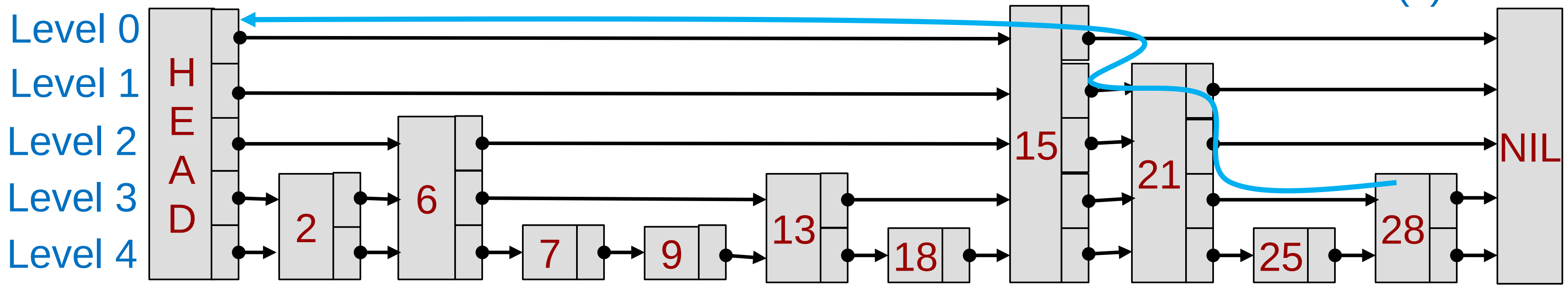


C(k) = 1+C(k-1)

Option B: No higher level, move left



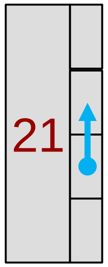
C(k) = 1+C(k)



Analysis

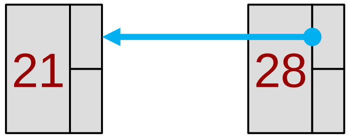
- Solve $C(k) = (1-p)(1+C(k)) + p(1+C(k-1))$
  - $C(k) = (1-p) + (1-p)C(k) + p + pC(k-1)$
  - $pC(k) = 1 + pC(k-1)$
  - $C(k) = 1/p + C(k-1)$
  - $= 1/p + 1/p + C(k-2)$
  - $= 1/p + 1/p + 1/p + C(k-3)$
  - $= k/p$
  - $= \log_2(N) / p = O(\log_2(N))$

Option A: If we can move up we do

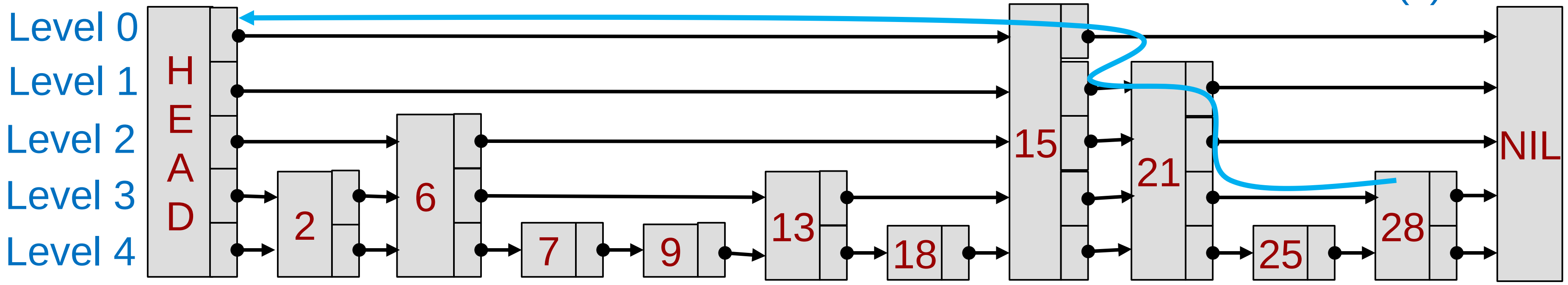


$C(k) = 1+C(k-1)$

Option B: No higher level, move left



$C(k) = 1+C(k)$

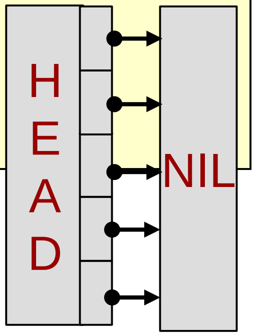


Node & Class Definition

- Each node has an array of "forward" ("next") pointers
- Head's key doesn't matter as we'll never compare it
- End's forward pointers don't matter since its key value is +INF

```
template < class K, class V >
struct SkipNode{
 K key;
 V value;
 SkipNode** forward; //array of ptrs

 SkipNode(K& k, V& v, int level){
   key = k;  value = v;
   forward = new SkipNode*[level+1];
 } };
```

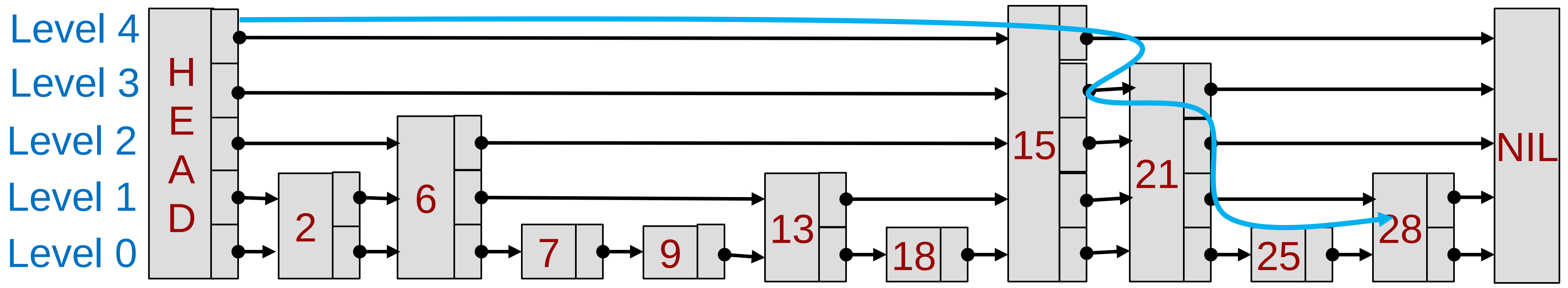
```
template < class K, class V >
class SkipList{
 int maxLevel;    // data members
 SkipNode* head;

 SkipList(int max){
   maxLevel = max;
   head = new SkipNode(dummy,dummy,maxLevel);
   SkipNode* end =
         new SkipNode(INFINITY,dummy,maxLevel);
   for(int i=0; i < maxLevel; i++){
     header->forward[i] = end;
  } }
};
```



Search Pseudocode

- search(28) would stop the for loop with current pointing at node 25, then take one more step

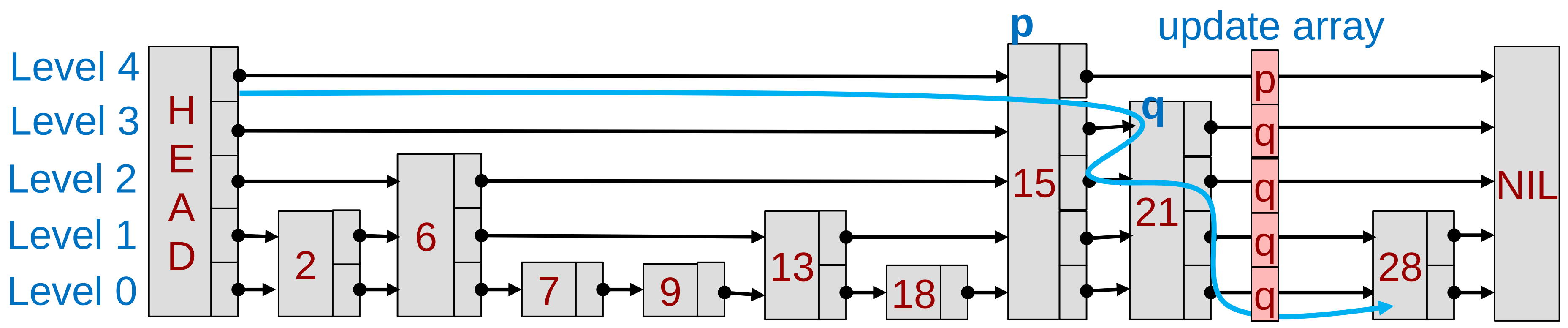
```
template < class K, class V >
SkipNode<K,V>* SkipList<K,V>::search(const Key& key){
  SkipNode<K,V>* current = head;
  for(int i=maxLevel; i >= 0; i--){
    while( current->forward[i]->key < key){
       current = current->forward[i];
    }
  }
  // will always stop on level 0 w/ current=node
  //  just prior to the actual target node or End node
  current = current->forward[0];
  if(current->key == key) return current;
  else return NULL;  // key is not in the list
}
```



Insert Pseudocode

- insert(25)
- As we walk we'll fill in an "update" array of the last nodes we walked through at each level since these will need to have their pointers updated

```
template < class K, class V >
void SkipList<K,V>::insert(const Key& key,
                           const Value& v){
  SkipNode<K,V>* current = head;
  vector<SkipNode<K,V>*> update(maxLevel+1);
  // perform typical search but fill in update array
  ...
  current = current->forward[0];
  if(current->key == key)
    { current->value = v; return; }
  else {
    int height = randomLevel();
    // Allocate new node, x
    for(int i=0; i < height; i++){
      x->forward[i] = update[i]->forward[i];
      update[i]->forward[i] = x;
}   }
```

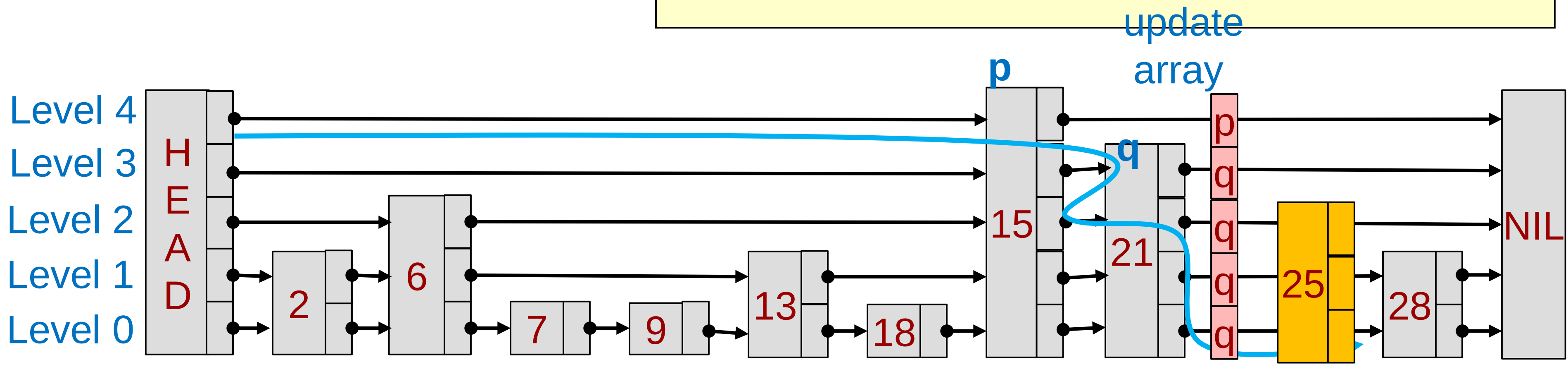


Insert Pseudocode

```
int SkipList<K,V>::randomLevel()
{
  int height = 1;
  // assume rand() returns double in range [0,1)
  while(rand() < p && height < maxLevel)
    height++;
  return height;
}
```

```
                lass K, class V >
                t<K,V>::insert(const Key& key,
                                    const Value& v){
                ,V>* current = head;
                pNode<K,V>*> update(maxLevel+1);
                 typical search but fill in update array

                current->forward[0];
                ->key == key)
      { current->value = v; return; }
  else {
    int height = randomLevel();
    // Allocate new node, x
    for(int i=0; i < height; i++){
      x->forward[i] = update[i]->forward[i];
      update[i]->forward[i] = x;
}   }
```

- randomLevel returns a height >h with probability $(1/p^h)$



Summary

- Skip lists are a randomized data structure
- Provide "expected" O(log(n)) insert, remove, and search
- Compared to the complexity of the code for structures like an RB-Tree they are fairly easy to implement
- In practice they perform quite well even compared to more complicated structures like balanced BSTs