# CSCI 104
# Amortized Analysis

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# A different form of runtime analysis

- Recall that a vector (from the STL) is implemented using an array.
- What is the worst-case runtime for the pushback function?
  - Is it $O(1)$?
  - If the array is full, we'll need to double the size of the array, which takes $\Theta(n)$ time!
  - It is correct to say that pushback takes worst-case $\Theta(n)$ runtime.
  - But, this analysis seems rather unfair, given that the worst-case will happen rarely, and at **predictable** intervals.

# Example

- You love going to Disneyland. You purchase an annual pass for $240. You visit Disneyland once a month for a year. Each time you go you spend $20 on food, etc.
  - What is the cost of a visit?
- Your annual pass cost is spread or "**amortized**" (or averaged) over the duration of its usefulness
- Often, an operation on a data structure will have similar "irregular" costs (i.e. if we can prove the worst case can't happen each call) that we can then amortize over future calls

# Amortized Runtime

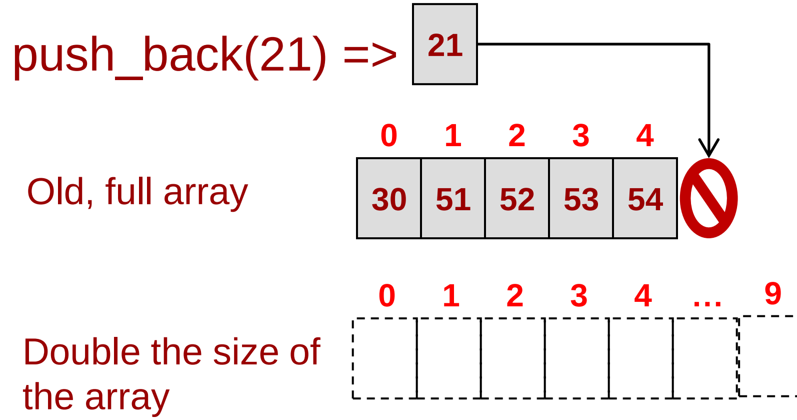
- We could accurately say that the average runtime for pushback is O(1).
  - This still doesn't capture everything: that implies that if we get bad luck, the average will be worse than O(1) [like a hash table]
  - There is no luck involved: we know exactly how many inputs will be required to produce the worst-case scenario, and it will always be the same effect.
- Amortized Runtime is a blend between average-case and worst-case. It is kind of the "worst-case average-case".
  - Use when it is provable that the worst-case runtime CAN'T happen on each call

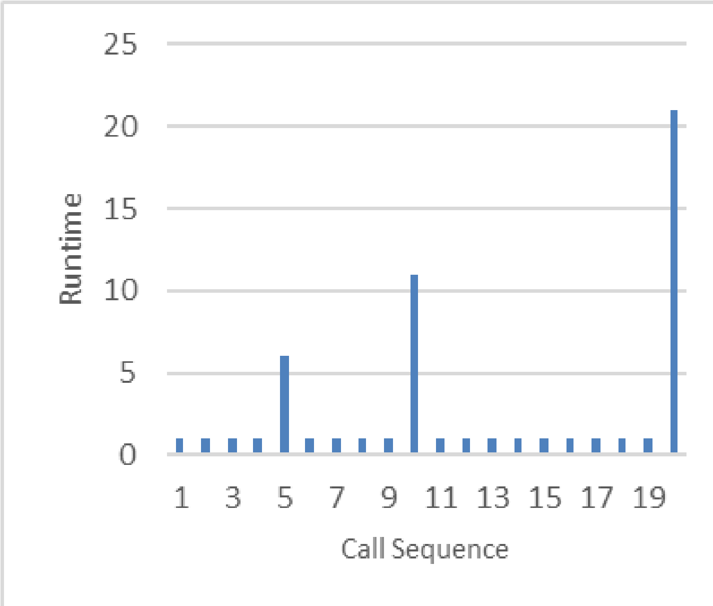
To use **amortized analysis**, usually some **state** must be maintained from one call to the next and that state will determine when the worst case happens.

# A LOOK BACK: AMORTIZED RUNTIME WITH VECTORS

# Amortized Run-time

- Used when it is impossible for the worst case of an operation to happen on each call (i.e. we can prove after paying a high cost that we will not have to pay that cost again for some number of future operations)
  - Example: Resizing a vector
- We will see 3 methods of performing amortized analysis



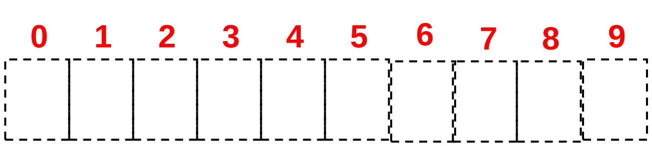



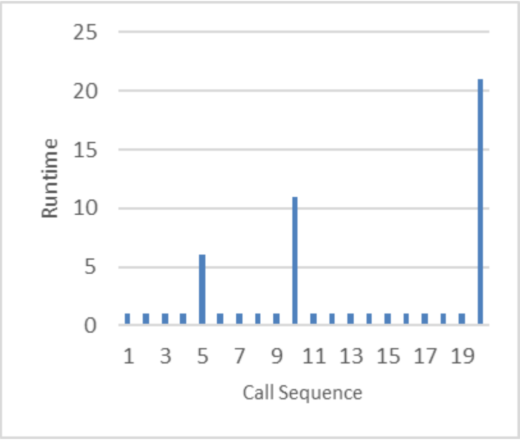
# Amortized Runtime

- **Method 1**: Analyze all k operations
- If in the worst case, the first k operations take a total (sum) of $\Theta$(m) time, then the average time per operation is $\frac{1}{k}\sum_k \theta(Operation_k) = \theta(\frac{m}{k})$.
  - The amortized runtime chooses the number and sequence of operations that produces the worst-possible average runtime.
  - It is like the "worst-case average-case".
  - Assume that the array starts at size 1, and you do n inserts. What is the amortized runtime for pushback?

# Pushback analysis, method 1

- Suppose we start at size 1 and double the array size when it becomes full
- There will be a few expensive pushbacks when we have to resize the array.
- When we have to resize and the array is of size, i, how costly is the pushback?
  - ________, where i is the current size of the array.
- If we started with an array size of 1, what values of i would cause us to resize: ________________________________
- How many expensive pushbacks will there be over n pushbacks?
  - ____________



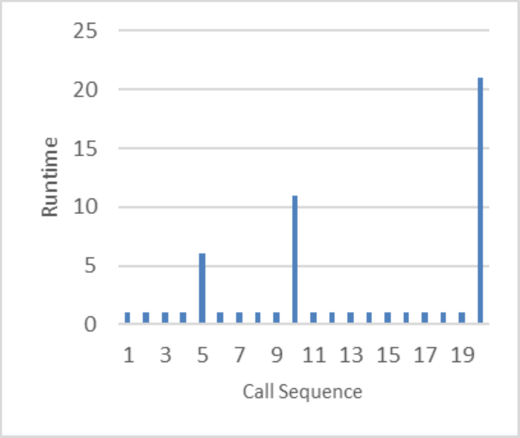
# Pushback analysis, method 1

- Suppose we start at size 1 and double the array size when it becomes full
- There will be a few expensive pushbacks, when we have to resize the array.
- When we have to resize and the array is of size, i, how costly is the pushback?
  - Θ(i), where i is the current size of the array.
- If we started with an array size of 1, what values of i would cause us to resize: 1, 2, 4, 8, 16, …
- How many expensive pushbacks will there be over n pushbacks?
  - log n
- The total runtime/cost is:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1} = \theta(c^n)$$

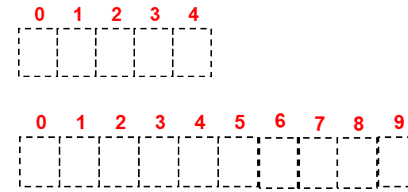
Pushbacks w/ resize.

$$\left(\sum_{i=1}^{\log n} 2^i\right) + \overbrace{(n - \log n)}^{\text{Pushbacks w/o resize.}} = \Theta(n)$$



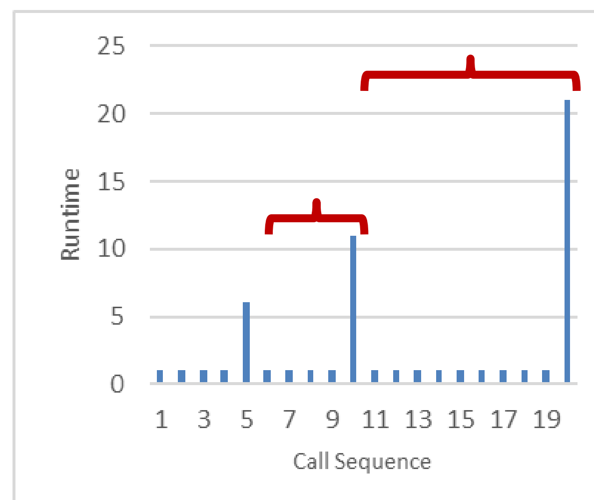

- The average is then $\frac{\Theta(n)\ total\ cost}{n\ calls} = \Theta(1)$
- So, the average time per operation is O(1). Guaranteed!

# Pushback analysis, Method 2




- **Method 2**: Analyze one "period/phase"
  - Let a new "phase" start just after the array has resized from n/2 to n.
- Analyze the amortized runtime for an arbitrary phase:
  - The array has just grown to size n, because we inserted $^n/_2 + 1$ things leaving n/2 – 1 free locations.
  - So we can insert $^n/_2 - 1$ more things in $\Theta(1)$ time.
  - On the next push back, we have to copy all n items to a new array (of size 2n), which takes Θ(n) time.
- Amortized runtime =

$$\frac{\overbrace{\left(^n/_2 - 1\right)\cdot 1}^{\text{Pushbacks w/o resize.}} + \overbrace{1}^{\text{1 Pushback w/ resize.}}\cdot \mathrm{n}}{^n/_2} = \Theta(1)$$



# Pushback analysis, Method 3

- **Method 3: Credit/Debit (Piggy Bank Method)**
- Again, let a new "phase" start just after the array has resized from n/2 to n.
- Every time we call pushback, we pay **3 dollars.**
  - Cheap operations only truly cost 1 dollar (to write the new value), so each of the cheap operations saves us a net of $2 which we place in a piggy bank.
  - When we get to an expensive operation, the last $n/2$ cheap operations have each paid 2 extra dollars for a total of n dollars saved up
  - We need to copy over the **n** elements to a new larger array, so we have one dollar for each item we need to copy.
  - We always have enough money saved up!
  - 3 dollars per pushback = $\Theta(1)$, so the amortized runtime is constant.

# Practice

- Let an integer, n, be represented as a Boolean array (requiring log(n) bits). You are given an **`increment()`** function
- What is the cost of incrementing the binary value?
- Each call to increment must visit the bits from right to left until we flip a bit from 0 to 1
- The runtime depends on how many bits we must visit
  - Some increments (from 1010 to 1011, for example) require only constant time.
  - Other increments (from 01111111 to 10000000) take a longer time.
- What is the worst-case runtime of our increment function?
  - $\Theta$(log n), all bits may need to flip in the worst case

**n**

| Bin. | Dec. |
|---|---|
| 0000 | 0 |
| 0001 | 1 |
| 0010 | 2 |
| 0011 | 3 |
| 0100 | 4 |
| 0101 | 5 |
| 0110 | 6 |
| 0111 | 7 |
| 1000 | 8 |
| 1001 | 9 |
| 1010 | 10 |
| 1011 | 11 |
| 1100 | 12 |
| 1101 | 13 |
| 1110 | 14 |
| 1111 | 15 |
| 0000 | 0 |

# Amortized analysis of the Binary Increment

- Starting at the least significant (rightmost) bit
  - If the current bit is a 0, we flip it and **stop**!
  - Otherwise, we flip the 1 to a 0 and continue to the next bit and **repeat**.
- Costs:
  - Define our "cost" as 1 unit for each bit we flip (i.e. every bit takes a single dollar to flip, from either 0 to 1 or 1 to 0)
  - We will always flip a single 0 to a 1.
  - We will flip a variable number of 1s to 0s.
- We will use the piggy bank method (method 3) to solve this.

```
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111
```

# Practice

- Recall: As stated, each time we call increment a single bit will flip from 0 to 1
- **Each time we call the increment function, we will pay a constant $2**
  - $1 for the bit that will flip from 0 to 1, and
  - $1 more in advance for when that bit eventually flips back to 0)
  - All of the bits start at 0.
  - Whenever we flip a bit from 0 to 1, we give both of our 2 dollars towards that bit. 1 dollar to cover the immediate costs, and the other dollar to be stored for when it eventually flips from 1 to 0.
  - A bit cannot flip from 1 to 0 if hadn't first flipped 0 to 1…so we'll never be in debt.
  - Since $2 = $\Theta(1)$, this takes amortized constant time!

| | Cost | Balance |
|---|---|---|
| 0000 | - | 0 |
| 0001 | 1 | 1 |
| 0010 | 2 | 1 |
| 0011 | 1 | 2 |
| 0100 | 3 | 1 |
| 0101 | 1 | 2 |
| 0110 | 2 | 2 |
| 0111 | 1 | 3 |
| 1000 | 4 | 1 |
| 1001 | 1 | 2 |
| 1010 | 2 | 2 |
| 1011 | 1 | 3 |
| 1100 | 3 | 2 |
| 1101 | 1 | 3 |
| 1110 | 2 | 3 |
| 1111 | 1 | 4 |
| 0000 | 4 | 2 |

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# An Alternate Approach – Expected Value

- We might also let **X** be a random variable defined to be *the number of bits that flip (i.e. cost of) on a call to increment* and compute **E[X]** (recall $\mathrm{E}[\mathrm{X}] = \sum_x x \cdot p(X = x)$ )
  - **X=1, P(X=1) = 1/2 or All calls cost >= 1**
  - **X=2, P(X=2) = 1/4 or 1/2 calls cost >= 2 (at least 1 more)**
  - **X=3, P(X=3) = 1/8 or 1/4 calls cost >= 3 (at least 1 more)**
  - **…**
  - $E[X] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \cdots + Last\ term \leq 2$
  - ***or***
  - $E[X] = 1 \cdot 1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{8} + \cdots \leq \sum_{i=0}^{\infty} \frac{1}{2^i} \leq 2$

| | |
|---|---|
| 0000 | - |
| 0001 | 1 |
| 0010 | 2 |
| 0011 | 1 |
| 0100 | 3 |
| 0101 | 1 |
| 0110 | 2 |
| 0111 | 1 |
| 1000 | 4 |
| 1001 | 1 |
| 1010 | 2 |
| 1011 | 1 |
| 1100 | 3 |
| 1101 | 1 |
| 1110 | 2 |
| 1111 | 1 |
| 0000 | 4 |