CSCI 104
Log Structured Merge Trees

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Series Summation Review

- Let n = 1 + 2 + 4 + … + 2^k = $\sum_{i=0}^{k} 2^i$. What is n?
  - n = $2^{k+1}-1$

- What is $\log_2(1) + \log_2(2) + \log_2(4) + \log_2(8)+\ldots+ \log_2(2^k)$
  = 0 + 1 + 2 + 3+… + k = $\sum_{i=0}^{k} i$
  - $O(k^2)$

Arithmetic series:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \theta(n^2)$$

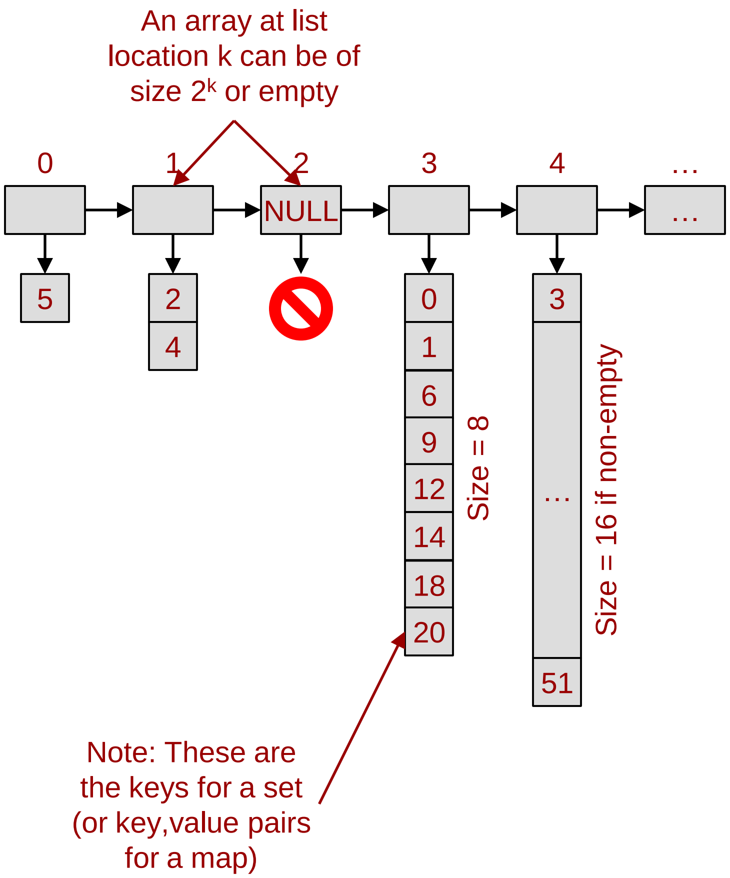
Geometric series

$$\sum_{i=1}^{n} c^i = \frac{c^{n+1}-1}{c-1} = \theta(c^n)$$

- So then what if k = log(n) as in:
  $\log_2(1) + \log_2(2) + \log_2(4) + \log_2(8)+\ldots+ \log_2(2^{\log(n)})$
  - $O(\log^2 n)$

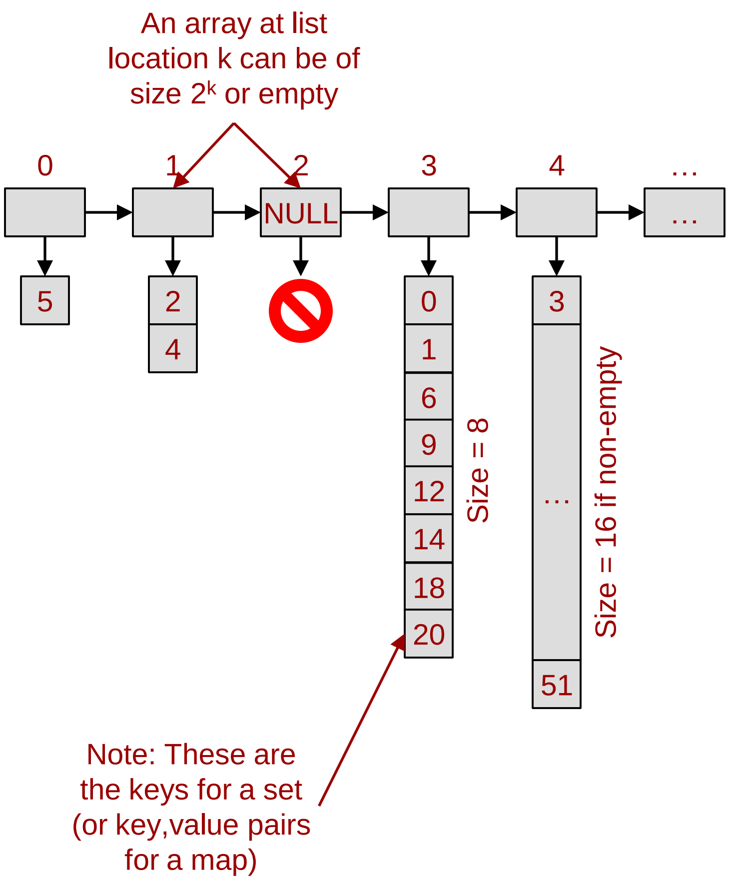
Merge Trees Overview

- Consider a list of (pointers to) arrays with the following constraints
  - Each array is sorted *though no ordering constraints exist between arrays*
  - The array at list index k is of exactly size 2^k or empty



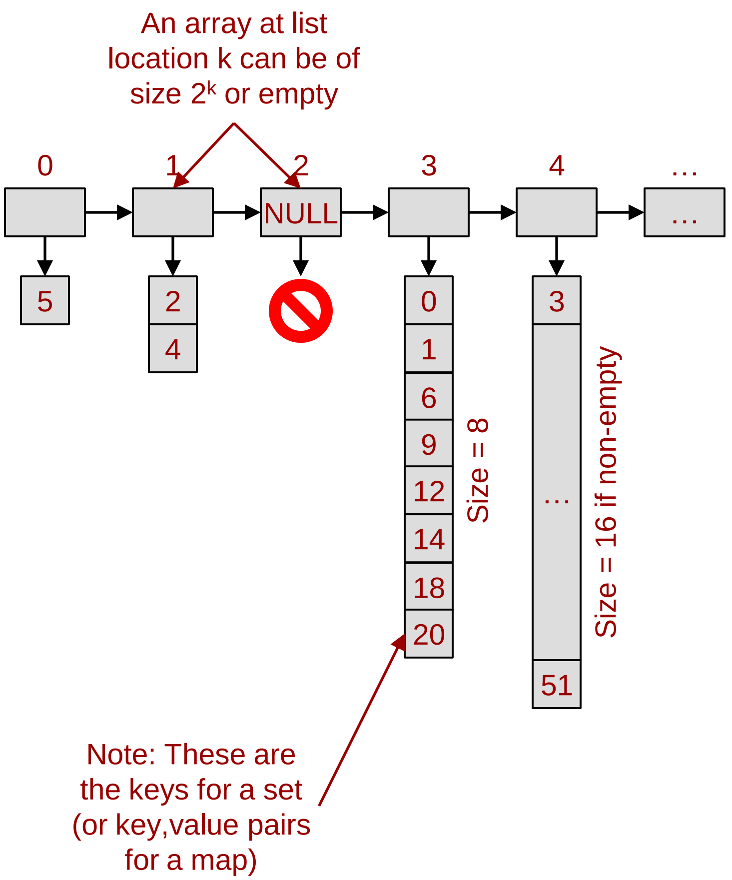
Merge Trees Size

- Define…
  - n as the # of keys in the entire structure
  - k as the size of the list (i.e. positions in the list)
- Given list of size k, how many total values, n, may be stored?
  - Let n = 1 + 2 + 4 + … + $2^{k-1} = \sum_{i=0}^{k-1} 2^i$. What is n?
- $n=2^k-1$



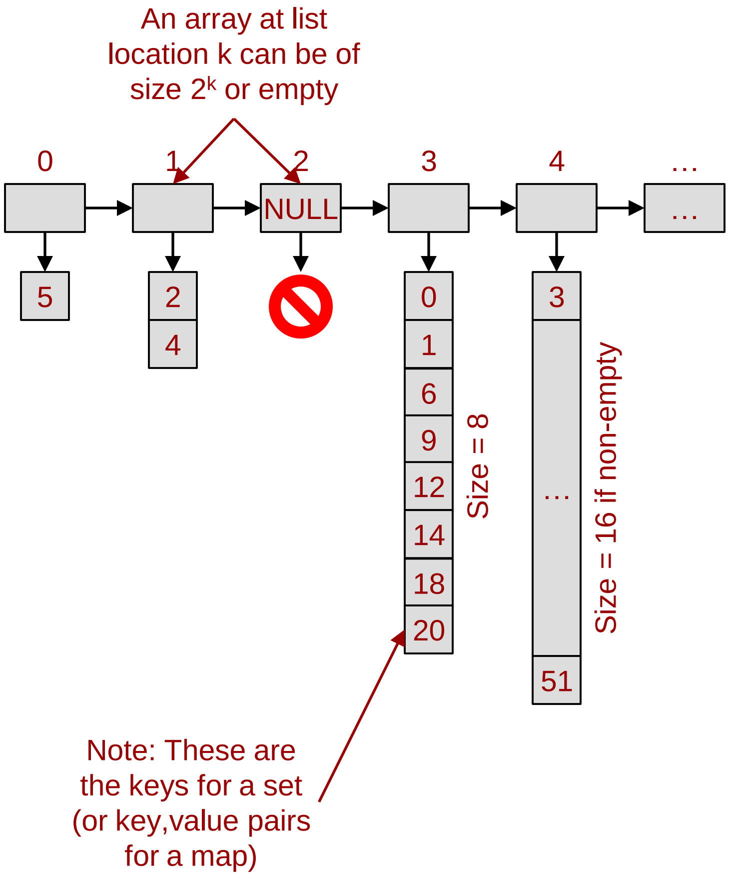
Merge Trees Find Operation

- To find an element (or check if it exists)
- Iterate through the arrays in order (i.e. start with array at list position 0, then the array at list position 1, etc.)
  - In each array perform a binary search
- If you reach the end of the list of arrays without finding the value it does not exist in the set/map



Find Runtime

- What is the worst case runtime of find?
  - When the item is not present which requires, a binary search is performed on each list
- $T(n) = \log_2(1) + \log_2(2) + \ldots + \log_2(2^{k-1})$
- $= 0 + 1 + 2 + \ldots + k-1 = \sum_{i=0}^{k-1} i$
  $= O(k^2)$
- But let's put that in terms of the number of elements in the structure (i.e. n)
  - Recall $k = \log_2(n+1)$
- So find is $O(\log_2(n)^2)$

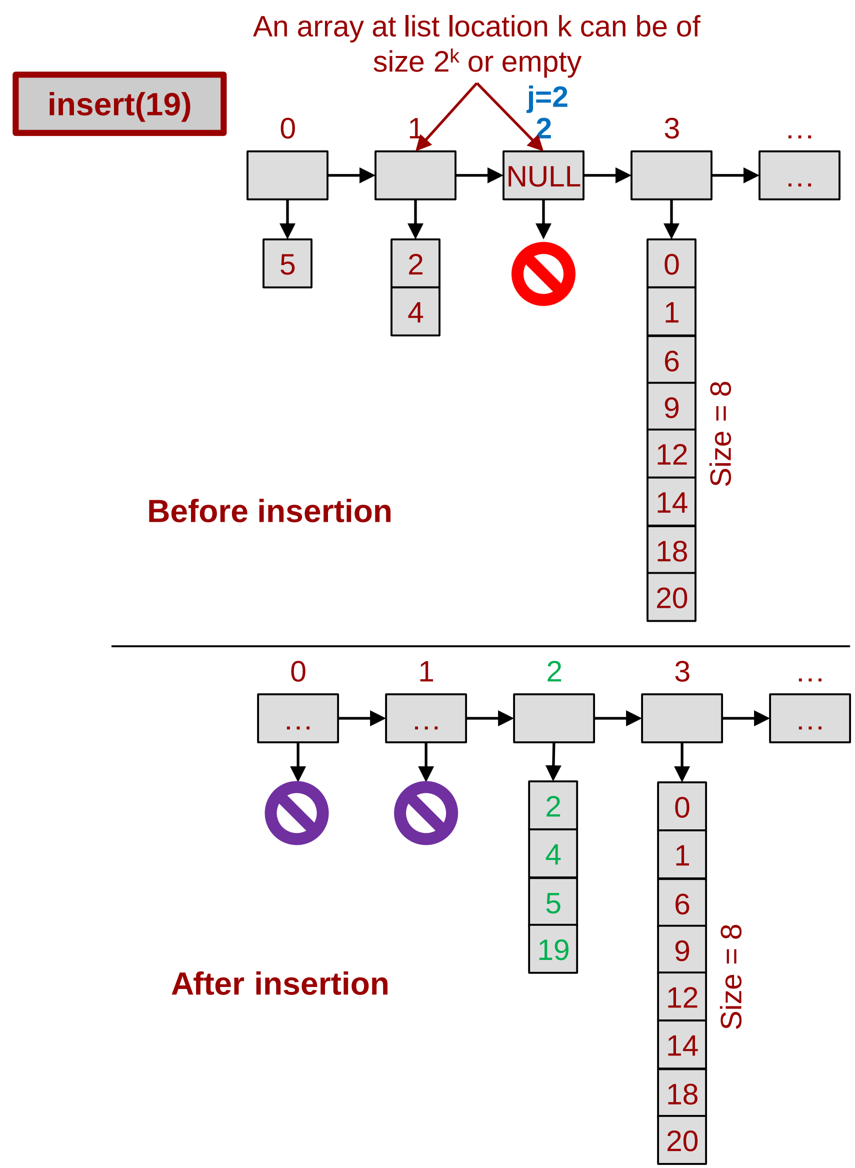


Improving Find's Runtime

- While we might be okay with $[\log(n)]^2$, how might we improve the find runtime in the general case?
  - Hint: I would be willing to pay O(1) to know if a key is not in a particular array without having to perform find
- A Bloom filter could be maintained alongside each array and allow us to skip performing a binary search in an array

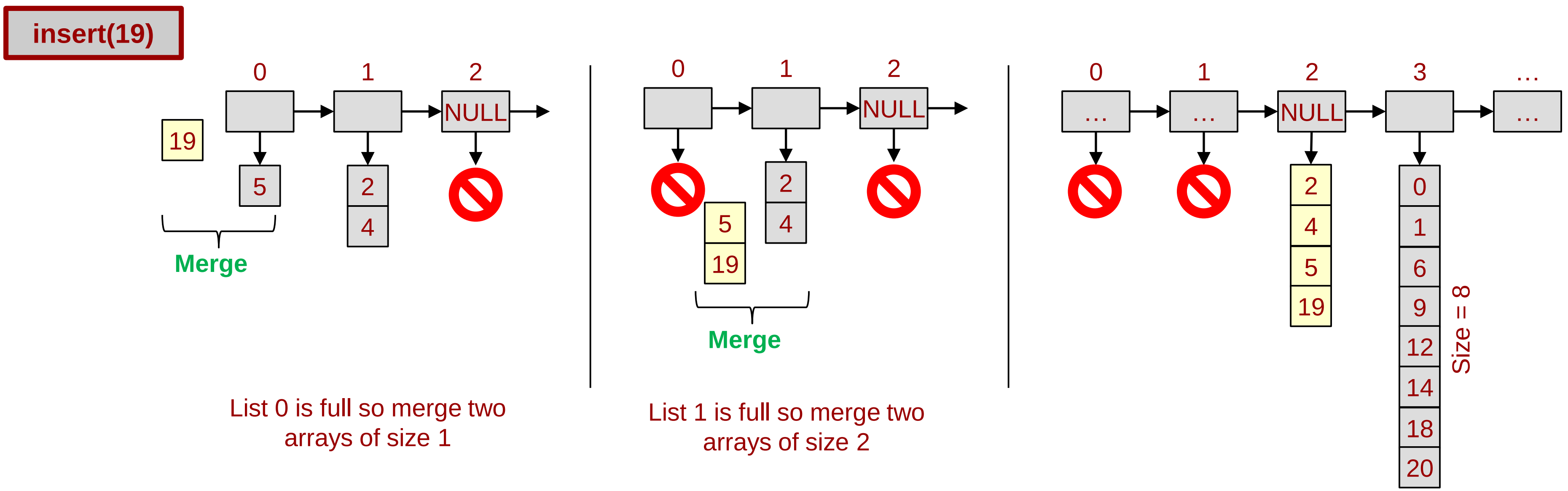
Insertion Algorithm

- Let j be the smallest integer such that array j is empty (first empty slot in the list of arrays)
- An insertion will cause
  - Location j's array to become filled
  - Locations 0 through j-1 to become empty



Insertion Algorithm

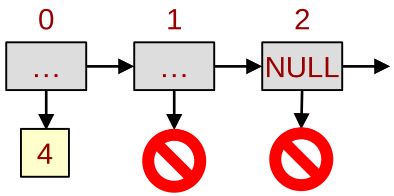
- Starting at array 0, iteratively merge the previously merged array with the next, stopping when an empty location is encountered



Insert Examples

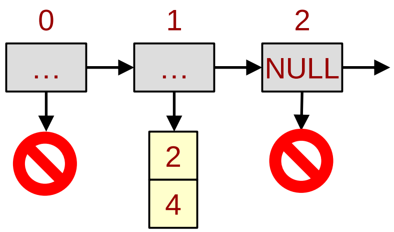
insert(4)

Cost = 1 /
Stop @ 0



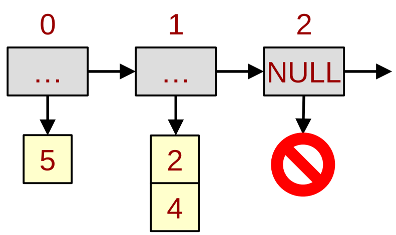
insert(2)

Cost = 3 /
Stop @ 1



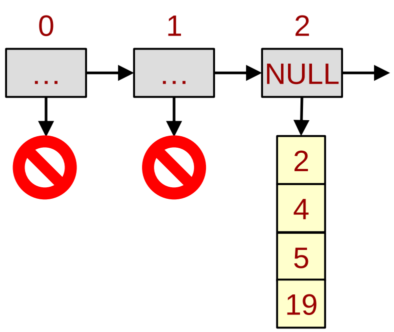
insert(5)

Cost = 1 /
Stop @ 0



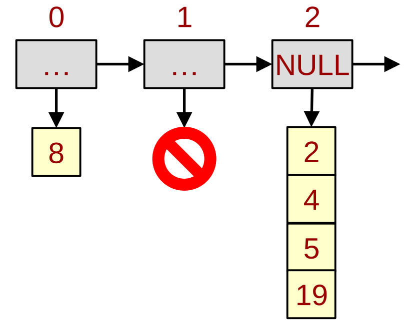
insert(19)

Cost = 7 /
Stop @ 2



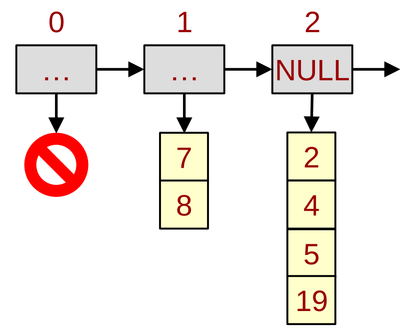
insert(8)

Cost = 1 /
Stop @ 0



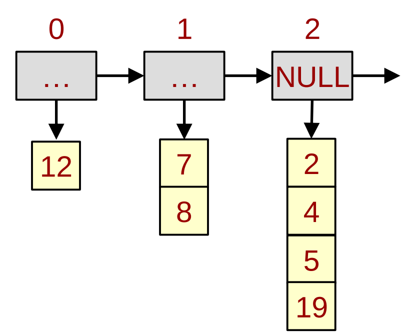
insert(7)

Cost = 1 /
Stop @ 1



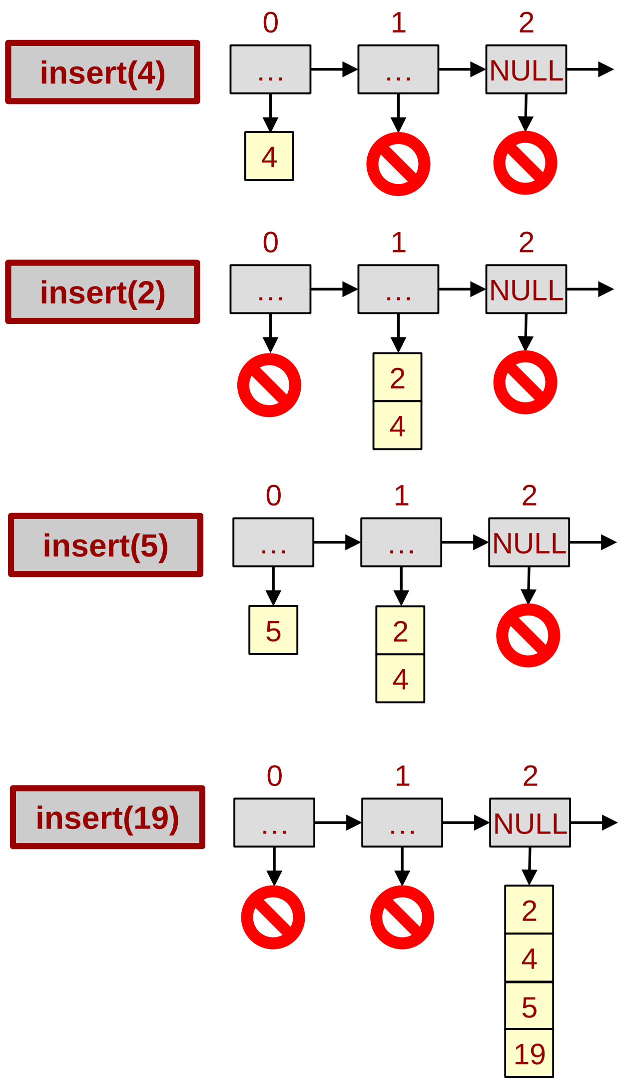
insert(12)

Cost = 1 /
Stop @ 0



Insertion Runtime: First Look

- Best case?
  - First list is empty and allows direct insertion in O(1)
- Worst case?
  - All list entries (arrays) are full so we have to merge at each location
  - In this case we will end with an array of size $n=2^k$ in position k
  - Also recall merging two sorted arrays of size m/2 is $\Theta(m)$
  - So the total cost of all the merges is $1 + 2 + 4 + 8 + \ldots + 2^k = \Theta(2^{k+1}) = \Theta(n)$
- But if the worst case occurs how soon can it occur again?
  - It seems the costs vary from one insert to the next
  - This is a good place to use amortized analysis

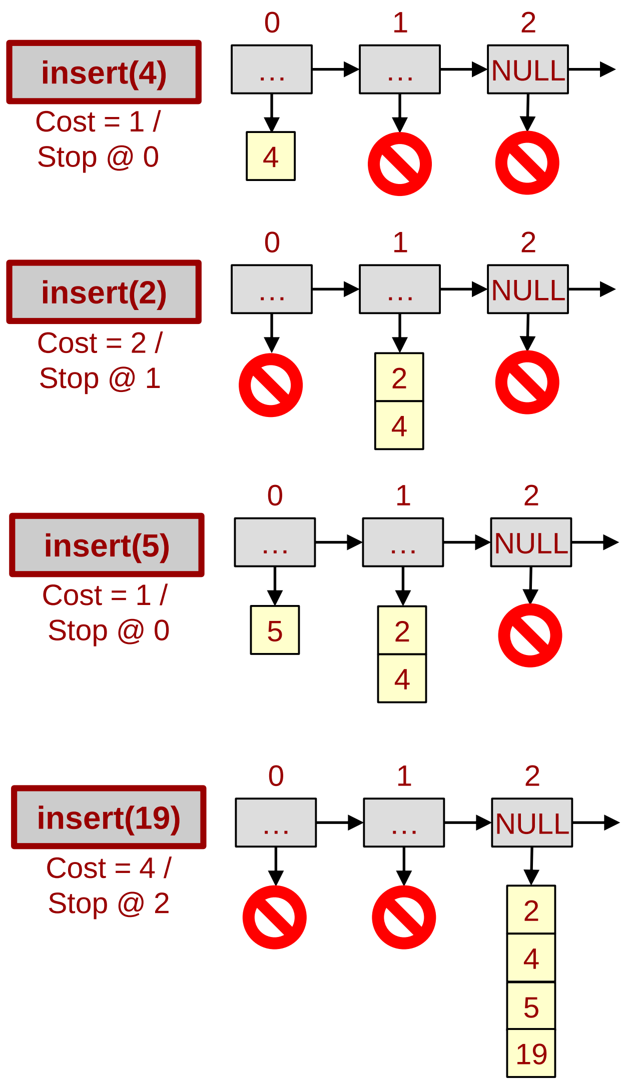


Total Cost for N insertions

- Reminder: Insert stopping at location k requires $1+2+4+\ldots+2^{k-1}+2^k = 2^{k+1}-1 = O(2^{k+1})$ merge steps
- Total cost of n=16 insertions:
  - Stop at: 0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4
  - Cost: $2^1+2^2+2^1+2^3+2^1+2^2+2^1+2^4+2^1+2^2+2^1+2^3+2^1+2^2+2^1+2^5$
- $=2^1*n/2 + 2^2*n/4 + 2^3*n/8 + 2^4*n/16 + 2^5*1$
- $= n + n + n + n + 2*n$
- $=n*\log_2(n) + 2n$
- Amortized cost = Total cost / n operations
  - $\log_2(n) + 2 = O(\log_2(n))$

Amortized Analysis of Insert

- We have said when you end (place an array) in position k you have to do $O(2^{k+1})$ work for all the merges
- How often do we end in position k
  - The 0^{th} position will be free with probability ½ (p=0.5)
  - We will stop at the 1^{st} position with probability ¼ (p=0.25)
  - We will stop at the 2^{nd} position with probability 1/8 (p=0.125)
  - We will stop at the k^{th} position with probability $1/2^{k+1} = 2^{-(k+1)}$
- So we pay 2^{k+1} with probability $2^{-(k+1)}$
- Suppose we have n items in the structure (i.e. max k is $\log_2 n$) what is the expected cost of inserting a new element
  - $\sum_{k=0}^{\log(n)} 2^{k+1} 2^{-(k+1)} = \sum_{k=0}^{\log(n)} 1 = \log(n)$



Summary

- Variants of log structured merge trees have found popular usage in industry
  - Starting array size might be fairly large (size of memory of a single server)
  - Large arrays (from merging) are stored on disk
- Pros:
  - Ease of implementation
  - Sequential access of arrays helps lower its constant factors
- Operations:
  - Find = $\log^2(n)$
  - Insert = Amortized $\log(n)$
  - Remove = often not considered/supported
- More reading:
  - http://www.benstopford.com/2015/02/14/log-structured-merge-trees/