CSCI 104
B-Trees (2-3, 2-3-4) and Red/Black Trees

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An example of B-Trees

2-3 TREES
Definition

• 2-3 Tree is a tree where
  – Non-leaf nodes have 1 value & 2 children or 2 values and 3 children
  – All leaves are at the same level
• Following the line of reasoning...
  – All leaves at the same level with internal nodes having at least 2 children implies a **full / complete** tree
    – FULL (Recall complete just means the lower level is filled left to right but not full)
  – A full tree with n nodes implies...
    • Height that is bounded by $\log_2(n)$
Implementation of 2- & 3-Nodes

- You will see that at different times 2 nodes may have to be upgraded to 3 nodes
- To model these nodes we plan for the worst case...a 3 node
- This requires wasted storage for 2 nodes

```cpp
template <typename T>
struct Item23 {
  T val1;
  T val2;
  Item23<T>* left;
  Item23<T>* mid;
  Item23<T>* right;
  bool twoNode;
};
```
2-3 Search Trees

• Similar properties as a BST

• 2-3 Search Tree
  – If a 2 Node with value, $m$
    • Left subtree nodes are $<$ node value
    • Right subtree nodes are $>$ node value
  – If a 3 Node with value, $l$ and $r$
    • Left subtree nodes are $< l$
    • Middle subtree $> l$ and $< r$
    • Right subtree nodes are $> r$

• 2-3 Trees are almost always used as search trees, so from now on if we say 2-3 tree we mean 2-3 search tree
2-3 Search Tree

• Binary search tree compared to 2-3 tree
• Check if 55 is in the tree?
2-3 Insertion Algorithm

- Key: Since all leaves must be at the same level ("leaves always have their feet on the ground"), insertion causes the tree to "grow upward"

- To insert a value,
  1. walk the tree to a leaf using your search approach
  2a. If the leaf is a 2-node (i.e. 1 value), add the new value to that node
  2b. Else break the 3-node into two 2-nodes with the smallest value as the left, biggest as the right, and median value promoted to the parent with smallest and biggest node added as children of the parent
  Repeat step 2(a or b) for the parent

- Insert 60, 20, 10, 30, 25, 50, 80

<table>
<thead>
<tr>
<th>Empty</th>
<th>Add 60</th>
<th>Add 20</th>
<th>Add 10</th>
<th>Add 30</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>60</td>
<td>20 60</td>
<td>10 20 60</td>
<td>10 30 60</td>
</tr>
</tbody>
</table>

Key: Any time a node accumulates 3 values, split it into single valued nodes (i.e. 2-nodes) and promote the median
2-3 Insertion Algorithm

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• Insert 60, 20, 10, 30, 25, 50, 80

Add 25

Add 50

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2-3 Insertion Algorithm

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- Insert 60, 20, 10, 30, 25, 50, 80

Key: Any time a node accumulates 3 values, split it into single valued nodes (i.e. 2-nodes) and promote the median

Add 80

10 25 50 60 30

20 30

10 25 50 60 80

20 30 60

10 25 50 80

20 60
2-3 Insertion Algorithm

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  – Repeat step 2(a or b) for the parent

• Insert 90, 91, 92, 93

Add 90

```
Add 90
```

```
30
  /|
 / |
20 60
 /|
/  |
10 25 50 80
```
2-3 Insertion Algorithm

- Key: Since all leaves must be at the same level, insertion causes the tree to "grow upward"

- To insert a value,
  1. walk the tree to a leaf using your search approach
  2a. If the leaf is a 2-node (i.e. 1 value), add the new value to that node
  2b. Else break the 3-node into two 2 nodes with the smallest value as the left, biggest as the right, and median value should be promoted to the parent with smallest and biggest node added as children of the parent
  Repeat step 2(a or b) for the parent

- Insert 90, 91, 92, 93
2-3 Insertion Algorithm

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  – 2b. Else break the 3-node into two 2 nodes with the smallest value as the left, biggest as the right, and median value should be promoted to the parent with smallest and biggest node added as children of the parent
  – Repeat step 2(a or b) for the parent

• Insert 90,91,92,93

Add 92

```
30
   /       \
  20       60  90
 /   \    /   \  \
10   25  50   80  91  92
```
2-3 Insertion Algorithm

- **Key:** Since all leaves must be at the same level, insertion causes the tree to "grow upward"

- **To insert a value,**
  1. walk the tree to a leaf using your search approach
  2a. If the leaf is a 2-node (i.e. 1 value), add the new value to that node
  2b. Else break the 3-node into two 2 nodes with the smallest value as the left, biggest as the right, and median value should be promoted to the parent with smallest and biggest node added as children of the parent
  - Repeat step 2(a or b) for the parent

- **Insert 90,91,92,93**

Add 93
Note

• 2-3 tree removal may be skipped due to time constraints
2-3 Tree Removal

• Key: 2-3 Trees must remain "full" (leaf nodes all at the same level)

• Remove
  – 1. Find data item to remove
  – 2. If data item is not in a leaf node, find in-order successor (which is in a leaf node) and swap values (it's safe to put successor in your location)
  – 3. Remove item from the leaf node
  – 4. If leaf node is now empty, call fixTree(leafNode)

• fixTree(n)
  – If n is root, delete root and return
  – Let p be the parent of n
  – If a sibling of n has two items
    • Redistribute items between n, sibling, and p and move any appropriate child from sibling to n
  – Else
    • Choose a sibling, s, of n and bring an item from p into s redistributing any children of n to s
    • Remove node n
    • If parent is empty, fixTree(p)

Another key: Want to get item to remove down to a leaf and then work up the tree
Remove Cases

Redistribute 1

\[ P \rightarrow L \]
\[ S \quad L \quad - \rightarrow S \quad S \quad P \]

Redistribute 2

\[ P \rightarrow L \]
\[ S \quad L \quad - \rightarrow S \quad P \quad a \quad b \quad c \quad d \]

Merge 1

\[ L \rightarrow \]
\[ S \quad - \rightarrow S \quad L \]

Merge 2

\[ P \rightarrow \]
\[ S \quad - \rightarrow S \quad P \quad a \quad b \quad c \]

Empty root

\[ - \rightarrow \]
\[ S \quad L \quad a \quad b \quad c \rightarrow S \quad L \quad a \quad b \quad c \]

\( P = \) parent
\( S = \) smaller
\( L = \) larger
Remove Examples

Remove 60

Remove 80

Key: Keep all your feet (leaves) on the ground (on the bottom row)

Not a leaf node so swap w/ successor at leaf

Since 2 items at leaf, just remove 60

Can't just delete because a 3-node would have only 2 children

Rotate 60 down into 50 to make a 3-node at the leaf and 2-node parent
Remove Cases

Redistribute 1

Redistribute 2

Merge 1

Merge 2

Empty root
Remove Examples

Remove 80

1. Remove root and thus height of tree decreases
2. Rotate parent down and empty node up, then recurse
3. Internal so swap w/ successor at leaf
4. Rotate parent down and empty node up, then recurse
Remove Cases

Redistribute 1

Redistribute 2

Merge 1

Merge 2

Empty root
Remove Exercise 1

Remove 30

Step 1: Not a leaf, so swap with successor

Step 2: Remove item from node

Step 3: Two values and 3 nodes, so merge. Must maintain levels.
Remove Exercise 1 (cont.)

Start over with the empty parent. Do another merge.

Step 4: Merge values

Step 5: Can delete the empty root node.
Remove Exercise 2

Remove 50

Step 1: It’s a leaf node, so no need to find successor. Remove the item from node.

Step 2: Since no 3-node children, push a value of parent into a child.

Step 3: Delete the node.
Remove Cases

Redistribute 1

Redistribute 2

Merge 1

Merge 2

Empty root
Insertion Exercise 1

Insert 12
Insertion Exercise 2

Insert 23
Insertion Exercise 3

Insert 39
Removal Exercise 4

Remove 10

10 28 33 37 50 75 95 105 120

40 100

25 30

70 90

110
Removal Exercise 5

Remove 40
Removal Exercise 6

Remove 30

30

20

70 90

10 15 25 50 75 95
Other Resources

• http://www.cs.usfca.edu/~galles/visualization/BTree.html
**Definition**

- 2-3-4 trees are very much like 2-3 trees but form the basis of a balanced, **binary** tree representation called Red-Black (RB) trees which are commonly used [used in C++ STL map & set]
  - We study them mainly to ease understanding of RB trees

- 2-3-4 Tree is a tree where
  - Non-leaf nodes have 1 value & 2 children or 2 values & 3 children or 3 values & 4 children
  - All leaves are at the same level

- Like 2-3 trees, 2-3-4 trees are always full and thus have an upper bound on their height of $\log_2(n)$
2-, 3-, & 4-Nodes

- 4-nodes require more memory and can be inefficient when the tree actually has many 2 nodes.
2-3-4 Search Trees

• Similar properties as a 2-3 Search Tree

• 4 Node:
  – Left subtree nodes are $< l$
  – Middle-left subtree $> l$ and $< r$
  – Right subtree nodes are $> r$
2-3-4 Insertion Algorithm

- **Key:** Rather than search down the tree and then possibly promote and break up 4-nodes on the way back up, split 4 nodes on the way down.

- **To insert a value,**
  - 1. If node is a 4-node
    - Split the 3 values into a left 2-node, a right 2-node, and promote the middle element to the parent of the node (which definitely has room) attaching children appropriately
    - Continue on to next node in search order
  - 2a. If node is a leaf, insert the value
  - 2b. Else continue on to the next node in search tree order

- **Insert 60, 20, 10, 30, 25, 50, 80**
2-3-4 Insertion Algorithm

- Key: Split 4 nodes on the way down
- To insert a value,
  - 1. If node is a 4-node
    - Split the 3 values into a left 2-node, a right 2-node, and promote the middle element to the parent of the node (which definitely has room) attaching children appropriately
    - Continue on to next node in search order
  - 2a. If node is a leaf, insert the value
  - 2b. Else continue on to the next node in search tree order

- Insert 60, 20, 10, 30, 25, 50, 80

Add 25

```
20
10  25  30  60
```

Add 50

```
20
10  25  30  60
```

Add 50

```
20
10  25  30  60
```

Split first, then add 50

```
10  25  50  60
```

Key: 4-nodes get split as you walk down thus, a leaf will always have room for a value
2-3-4 Insertion Algorithm

- Key: Split 4 nodes on the way down
- To insert a value,
  - 1. If node is a 4-node
    - Split the 3 values into a left 2-node, a right 2-node, and promote the middle element to the parent of the node (which definitely has room) attaching children appropriately
    - Continue on to next node in search order
  - 2a. If node is a leaf, insert the value
  - 2b. Else continue on to the next node in search tree order
- Insert 60, 20, 10, 30, 25, 50, 80

Add 80

Key: 4-nodes get split as you walk down thus, a leaf will always have room for a value
2-3-4 Insertion Exercise 1

Add 55

- 20 30
- 10 25 50 60 80
2-3-4 Insertion Exercise 2

Add 58

Diagram: A 2-3-4 tree with nodes 10, 25, 50, 55, 80, 20, 30, 60.
2-3-4 Insertion Exercise 3

Add 57

30

20 60

10 25 50 55 58 80
2-3-4 Insertion Exercise 3

Resulting Tree

```
     30
    /  \
   20   55 60
  /     /  \
10  25 50 57 58 80
```
B-Trees

- 2-3-4 trees are just instances of a more general data structure known as B-Trees
- Define minimum number of children (degree) for non-leaf nodes, \( d \)
  - Non-root nodes must have at least \( d-1 \) keys and \( d \) children
  - All nodes must have at most \( 2d-1 \) keys and \( 2d \) children
  - 2-3-4 Tree (\( d=2 \))
- Used for disk-based storage and indexing with large value of \( d \) to account for large random-access lookup time but fast sequential access time of secondary storage
B Tree Resources

• https://www.cs.usfca.edu/~galles/visualizatio
  n/BTree.html

• http://ultrastudio.org/en/2-3-4_tree
"Balanced" Binary Search Trees

RED BLACK TREES
Red Black Trees

- A red-black tree is a binary search tree
  - Only 2 nodes (no 3- or 4-nodes)
  - Can be built from a 2-3-4 tree directly by converting each 3- and 4- nodes to multiple 2-nodes
- All 2-nodes means no wasted storage overheads
- Yields a "balanced" BST
- "Balanced" means that the height of an RB-Tree is at MOST twice the height of a 2-3-4 tree
  - Recall, height of 2-3-4 tree had an upper bound of $\log_2(n)$
  - Thus height or an RB-Tree is bounded by $2*\log_2{n}$ which is still $O(\log_2(n))$
Red Black and 2-3-4 Tree Correspondence

• Every 2-, 3-, and 4-node can be converted to...
  – At least 1 black node and 1 or 2 red children of the black node
  – Red nodes are always ones that would join with their parent to become a 3- or 4-node in a 2-3-4 tree

S = Small
M = Median
L = Large
Red Black Trees

- Below is a 2-3-4 tree and how it can be represented as a directly corresponding RB-Tree.
- Notice at most each 2-3-4 node expands to 2 level of red/black nodes.
- **Q:** Thus if the height of the 2-3-4 tree was bound by $\log_2 n$, then the height of an RB-tree is bounded by?
- **A:** $2 \cdot \log_2 n = O(\log_2 n)$

![Image](image.png)
Red-Black Tree Properties

• Valid RB-Trees maintain the invariants that...

• 1. No path from root to leaf has two consecutive red nodes (i.e. a parent and its child cannot both be red)
  – Since red nodes are just the extra values of a 3- or 4-node from 2-3-4 trees you can't have 2 consecutive red nodes

• 2. Every path from leaf to root has the same number of black nodes
  – Recall, 2-3-4 trees are full (same height from leaf to root for all paths)
  – Also remember each 2, 3-, or 4- nodes turns into a black node plus 0, 1, or 2 red node children

• 3. At the end of an operation the root should always be black

• 4. We can imagine leaf nodes as having 2 non-existent (NULL) black children if it helps
Red-Black Insertion

• Insertion Algorithm:
  – 1. Insert node into normal BST location (at a leaf location) and color it RED
  – 2a. If the node's parent is black (i.e. the leaf used to be a 2-node) then DONE (i.e. you now have what was a 3- or 4-node)
  – 2b. Else perform fixTree transformations then repeat step 2 on the parent or grandparent (whoever is red)

• fixTree involves either
  – recoloring or
  – 1 or 2 rotations and recoloring

• Which case of fixTree you perform depends on the color of the new node's "aunt/uncle"
fixTree Cases

1. Recolor
   - Origin: 
     - P
     - U
     - N
   - Target: 
     - P
     - U
     - N
   - Note: For insertion/removal algorithm we consider non-existent leaf nodes as black nodes

2. Recolor
   - Origin: 
     - P
     - U
     - N
   - Target: 
     - P
     - U
     - N

3. Recolor Root
   - Origin: 
     - R
   - Target: 
     - R

Note: For insertion/removal algorithm we consider non-existent leaf nodes as black nodes
fixTree Cases

4. 1 Rotate / Recolor

Right rotate of P,G

5. 2 Rotates / Recolor

Left rotate of N,P

Right rotate of N,G & Recolor

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Insertion

- Insert 10, 20, 30, 15, 25, 12, 5, 3, 8

Empty | Insert 10 | Insert 20 | Insert 30 | Case 4: Left rotate and recolor

10

10

10

10

20

20

20

30

20

10

30

Insert 15

Case 2: Recolor

10

15

20

30

Insert 25

10

15

20

30

25
Insertion

• Insert 10, 20, 30, 15, 25, 12, 5, 3, 8

Insert 12

Case 5: Right Rotate...

Insert 5

Case 1: Recolor

Recursive call "fix" on 12 but it's parent is black so we're done
Insertion

- Insert 10, 20, 30, 15, 25, 12, 5, 3, 8

Insert 3

Case 4: Rotate
Insertion

• Insert 10, 20, 30, 15, 25, 12, 5, 3, 8
Insertion Exercise 1

Insert 27

G

P

N 27
Insertion Exercise 1

This is case 5.
1. Left rotate around P
2. Right rotate around N
3. Recolor
Insertion Exercise 2

Insert 40
Aunt and Parent are the same color. So recolor aunt, parent, and grandparent.
Insertion Exercise 2

Aunt and Parent are the same color. So recolor aunt, parent, and grandparent.
Insertion Exercise 3

Insert 50

G

P

N
Insertion Exercise 3

Insert 50

Remember, empty nodes are black.
Do a left rotation around P and recolor.
Insertion Exercise 4

Insert 45

12
  5
  3
  10
  15
  20
  27
  25
  40
  8
  45

A
G
P
N
Aunt and Parent are the same color. Just recolor.
Insertion Exercise 4
Final Result
Insertion Exercise 5

Insert 9

12
/
5
/
3

G
/
10
/
8

P
/
9

A
/
20
/
15
/
15
/
5
/
40
/
30
/
50

N
/
25
/
45
Insertion Exercise 5
RB-Tree Visualization & Links

RB TREE IMPLEMENTATION
Hints

• Implement private methods:
  – findMyUncle()
  – AmIaRightChild()
  – AmIaLeftChild()
  – RightRotate
  – LeftRotate

• Need to change x's parent, y's parent, b's parent, x's right, y's left, x's parent's left or right, and maybe root
Hints

• You have to fix the tree after insertion if...
• Watch out for traversing NULL pointers
  – node->parent->parent
  – However, if you need to fix the tree your grandparent...
• Cases break down on uncle's color
  – If an uncle doesn't exist (i.e. is NULL), he is (color?)...

![Diagram](image-url)
### fixTree Cases

1. **Recolor**

   - Original Tree:
     - **G**
     - **P**
     - **U**
     - **N**
     - **c**
     - **a**
     - **b**
   - Recolored Tree:
     - **G**
     - **P**
     - **U**
     - **N**
     - **c**
     - **a**
     - **b**

2. **Recolor**

   - Original Tree:
     - **G**
     - **P**
     - **U**
     - **N**
     - **c**
     - **a**
     - **b**
   - Recolored Tree:
     - **G**
     - **P**
     - **U**
     - **N**
     - **c**
     - **a**
     - **b**

3. **Recolor Root**

   - Original Tree:
     - **R**
   - Recolored Tree:
     - **R**

**Note:** For insertion/removal algorithm we consider non-existent leaf nodes as black nodes.
fixTree Cases

4.

1 Rotate / Recolor

Right rotate of P,G

2 Rotates / Recolor

Left rotate of N,P

Right rotate of N,G & Recolor

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