CSCI 104
Recursion –
Combinations & Backtracking

Mark Redekopp

Aaron Cote’

Recursion in CS 104

- Problem in which the solution can be expressed in terms of itself (usually a smaller instance/input of the same problem) ***and a base/terminating case***
- Recursion is a **key concept** in this course
  - But it rarely comes easily to students. You must work at it!
- Many problems that would be VERY difficult to solve without recursion (i.e. only loops) have extremely ***elegant solutions*** to problems
  - Learn to look for those elegant solutions
  - In this class, assume the recursive approach has an elegant/simple solution
  - If you find yourself writing a large, complex recursive solution, assume you are doing something you should not!
    - Stop and reconsider how it should be done

Simple vs. Multiple Recursion

- "**Simple**" recursion refers to functions that contain just **ONE recursive call**
  - Can be head or tail recursion (explained soon)
  - **Can easily be replaced by a loop**
- The power of recursion usually comes when the function makes **2 OR MORE recursive calls** (aka "**multiple recursion**")
  - Elegant recursive solutions that would be **MUCH harder to implement iteratively** (usually need a separate stack data structure)
- We'll focus on multiple recursion

```
void print(Item* p)
{
  if(p == NULL) return;
  else {
    cout << p->val << endl;
    print(p->next);
  }
}
```

Simple Recursion
(1 recursive call)

```
void postorder(TNode* t)
{  if(t == NULL) return
   postorder(t->left)
   postorder(t->right)
   process(t) // print val.
}
```

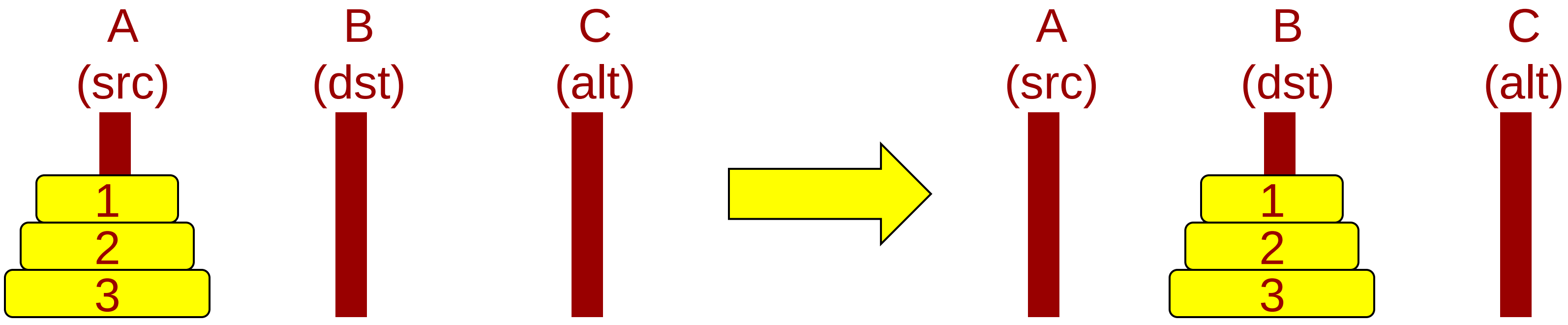
Multiple Recursion
(2 or more recursive calls)

Steps to Formulating Recursive Solutions

1. Solve a few instances of the problem to discover the recursive structure
2. Identify how the problem can be decomposed into smaller problems of the same form
   - Does solving the problem on an input of smaller value or size help formulate the solution to the larger
3. Identify the base case
   - An input for which the answer is trivial
4. Assume the recursive call for the smaller problem "magically" computes the correct solution(s) to those problem(s) and **identify how to combine those solution(s)** from the smaller problem(s) into the solution for the larger problem

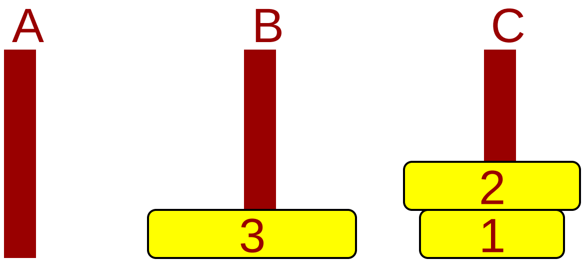
Towers of Hanoi Problem

- Problem Statements: Move n discs from source pole to destination pole (with help of a 3rd alternate pole)
  - Cannot place a larger disc on top of a smaller disc
  - Can only move one disc at a time



Start (n=3)

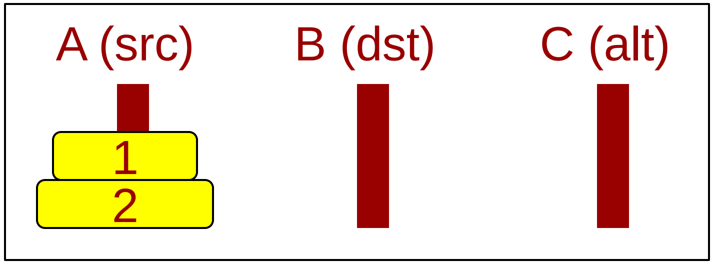
Goal (n=3)



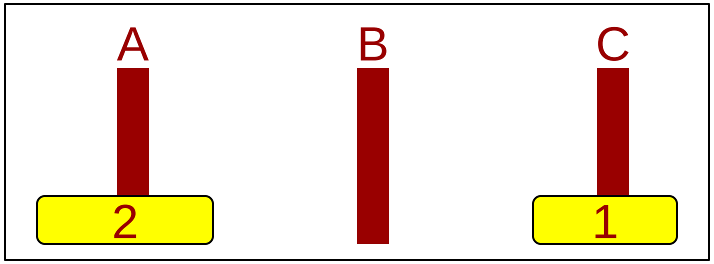
Not allowed

Observation 1

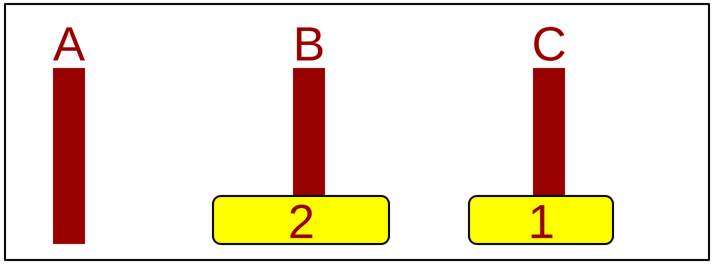
- Observation 1: Disc 1 (smallest) can always be moved
- Solve the n=2 case:



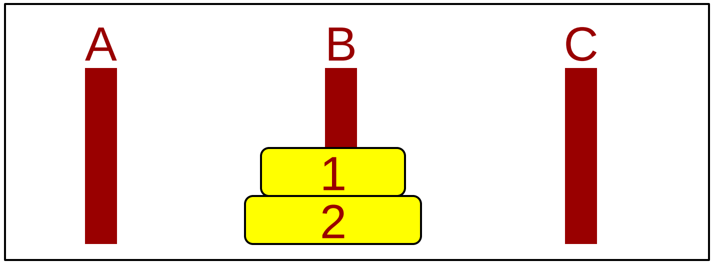
Start



Move 1 from src to alt



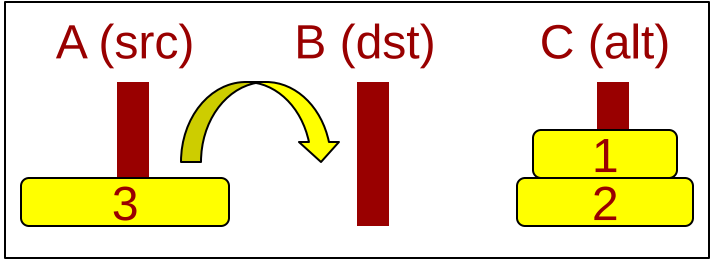
Move 2 from src to dst

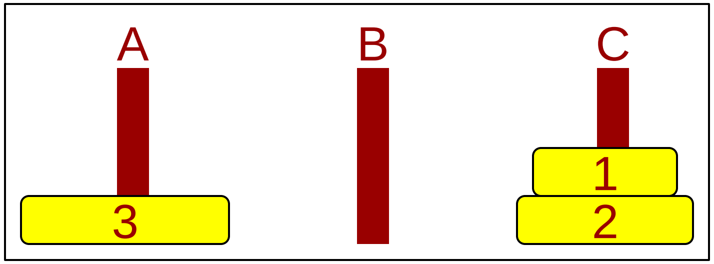


Move 1 from alt to dst

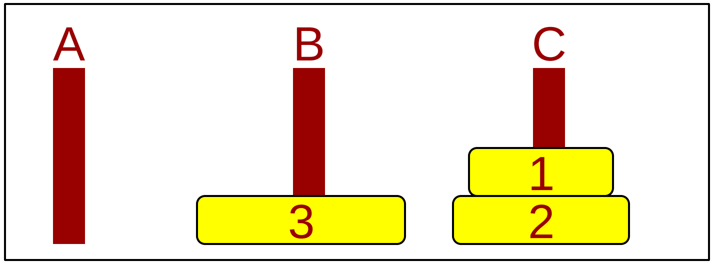
Observation 2

- Observation 2: If there is only one disc on the src pole and the dest pole can receive it the problem is trivial

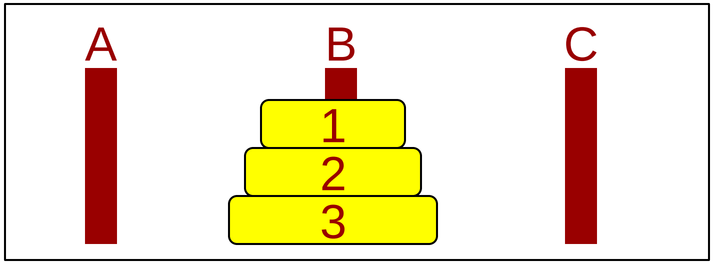




Move n-1 discs from src to alt



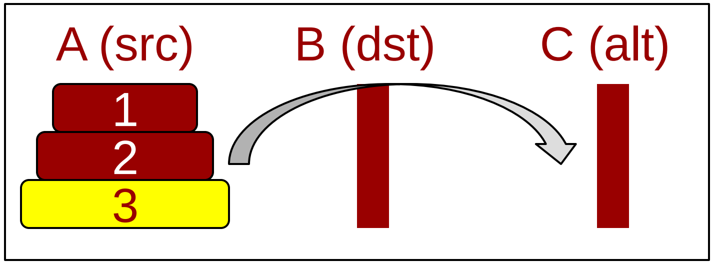
Move disc n from src to dst



Move n-1 discs from alt to dst

Recursive solution

- But to move n-1 discs from src to alt is really a smaller version of the same problem with
  - n => n-1
  - src=>src
  - alt =>dst
  - dst=>alt

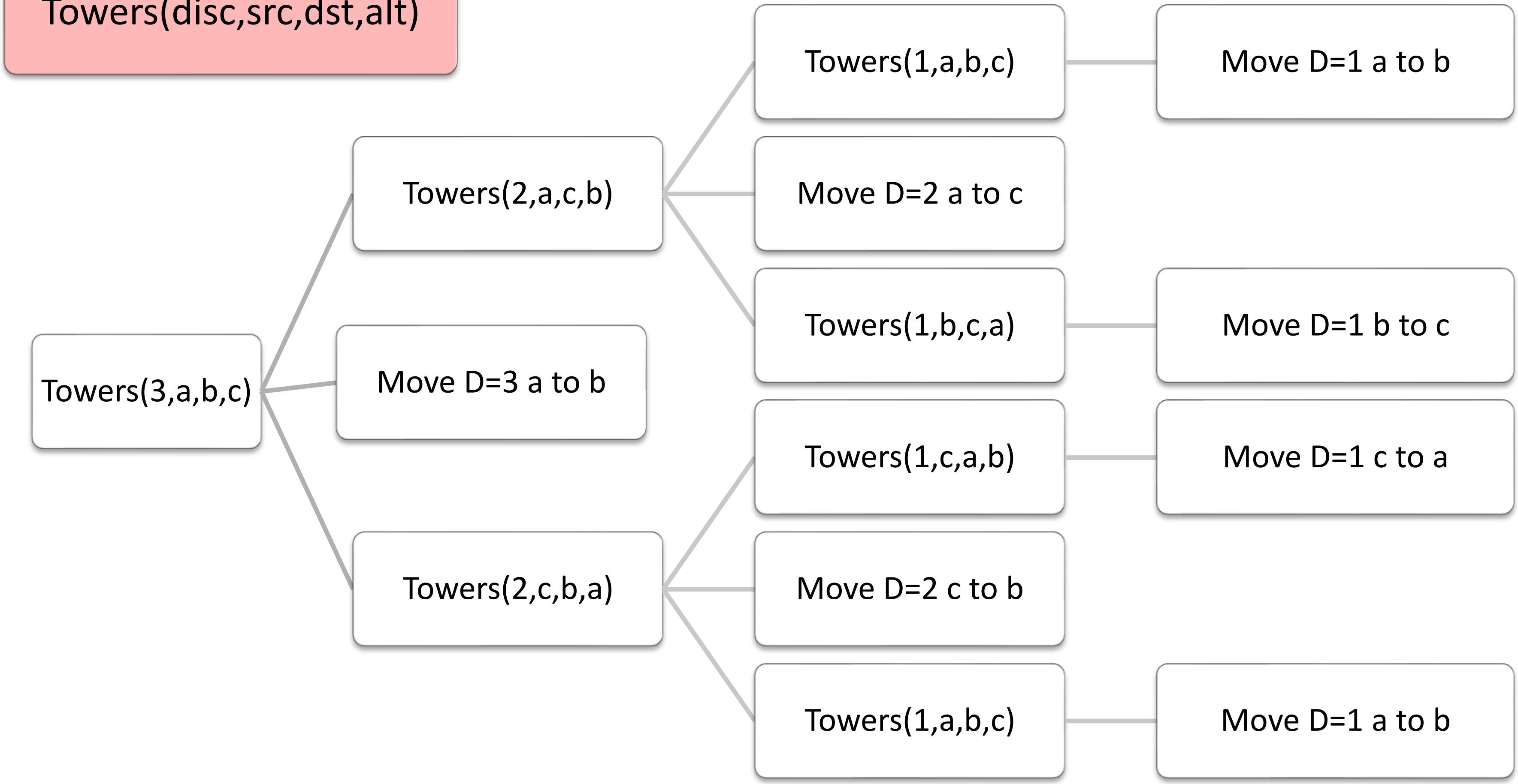


- Towers(n,src,dst,alt)
  - Base Case: n==1 // Observation 1: Disc 1 always movable
    - Move disc 1 from src to dst
  - Recursive Case: // Observation 2: Move of n-1 discs to alt & back
    - Towers(n-1,src,alt,dst)
    - Move disc n from src to dst
    - Towers(n-1,alt,dst,src)

Recursive Box Diagram

Towers Function Prototype

Towers(disc,src,dst,alt)



Towers(3,a,b,c)

Towers(2,a,c,b)

Move D=3 a to b

Towers(2,c,b,a)

Towers(1,a,b,c)

Move D=2 a to c

Towers(1,b,c,a)

Towers(1,c,a,b)

Move D=2 c to b

Towers(1,a,b,c)

Move D=1 a to b

Move D=1 b to c

Move D=1 c to a

Move D=1 a to b

GENERATING ALL COMBINATIONS

Recursion's Power

- The power of recursion often comes when each function instance makes ***multiple*** recursive calls
- As you will see this often leads to an exponential number of "combinations" being generated/explored in an easy fashion

Binary Combinations

- If you are given the value, n, and a string with n characters could you generate all the combinations of n-bit binary?
- Do so recursively!

Exercise: bin_combo_str

```
0
1
```

1-bit Bin.

```
00
01
10
11
```

2-bit Bin.

```
000
001
010
011
100
101
110
111
```

3-bit Bin.

```
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111
```

4-bit Bin.

Recursion and DFS

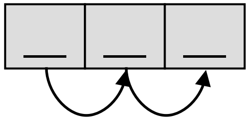
- Recursion forms a kind of Depth-First Search

Options

0
1

Generally: Recursion must perform the same code sequence for each item. Where we need variation, use 'if' statements.

N = length

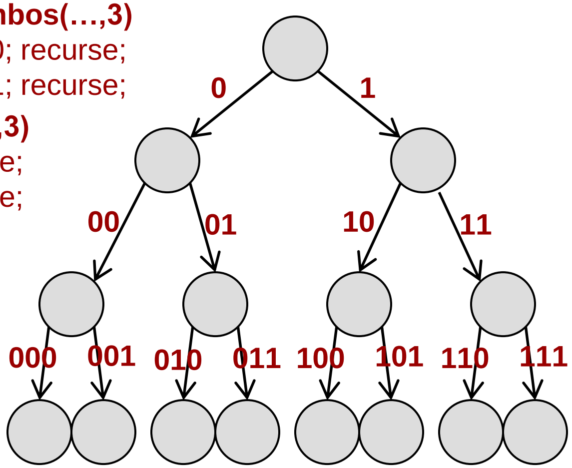


binCombos(…,3)
Set to 0; recurse;
Set to 1; recurse;

binCombos(…,3)
Set to 0; recurse;
Set to 1; recurse;

binCombos(…,3)
Set to 0; recurse;
Set to 1; recurse;

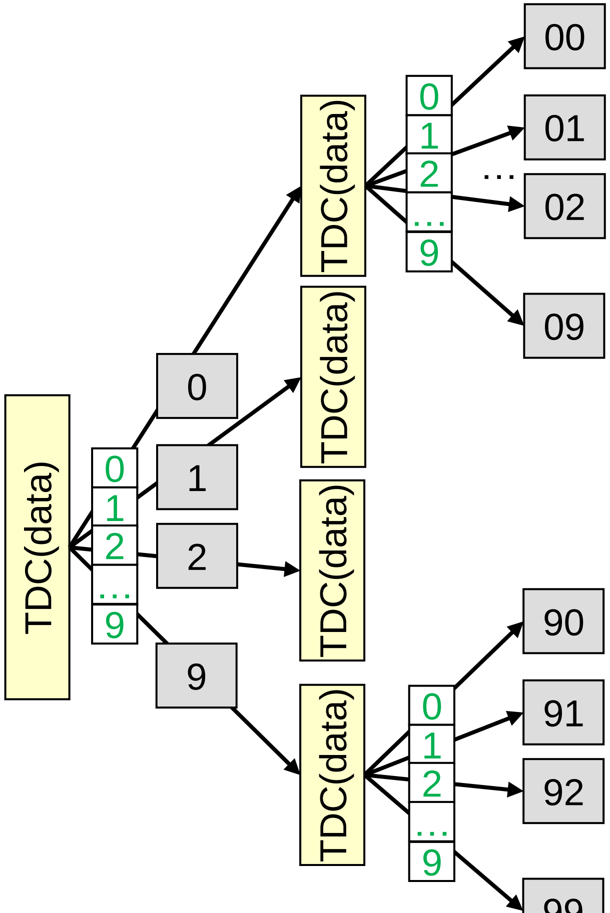
binCombos(…,3)
Base case



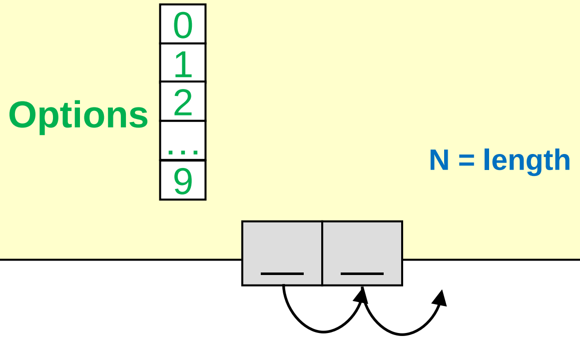
```
// user interface
void binCombos(int len)
{
  binCombos("", len);
}

// helper-function
void binCombos(string prefix,
               int len)
{
  if(prefix.length() == len )
    cout << prefix << endl;
  else {
    // recurse
    binCombos(_____________, len);
    // recurse
    binCombos(_____________, len);
  }
}
```

Generating All Combinations

- Recursion offers a simple way to generate all N-length combinations of from a set of options, S
  - Example: Generate all 2-digit decimal numbers (N=2, S={0,1,…,9})



```
void NDigDecCombos(string data, int n)
{
  if(data.size() == n )
    cout << data;
  else {
    for(int i=0; i < 10; i++){
     // recurse
     NDigDecCombos(data+(char)('0'+i),n);
    }
  }
}
```



Another Exercise

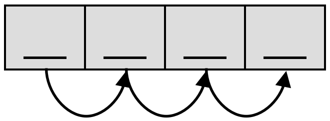
- Generate all string combinations of length n from a given list (vector) of characters

```
#include <iostream>
#include <string>
#include <vector>
using namespace std;

void all_combos(vector<char>& letters, int n)
{
  // ???
}

int main() {
   vector<char> letters = {'U', 'S', 'C'};

   all_combos(letters, 4);

   return 0;
}
```

Options

U
S
C

N = length



Use recursion to walk down the 'places'

At each 'place' iterate through & try all options

Recursion and Combinations

- Recursion provides an elegant way of generating all **n**-length combinations of a set of values, S.
  - Ex. Generate all length-**n** combinations of the letters in the set **S**={'U','S','C'} (i.e. for n=2: UU, US, UC, SU, SS, SC, CU, CS, CC)
- General approach:
  - Need some kind of **array/vector/string** to store partial answer as it is being built
  - Each recursive call is only responsible for one of the **n** "places" (say location, **i**)
  - The function will iteratively (loop) try each option in **S** by setting location i to the current option, then recurse to handle all remaining locations (i+1 to n)
    - Remember you are responsible for only one location
  - Upon return, try another option value and recurse again
  - Base case can stop when all n locations are set (i.e. recurse off the end)
  - Recursive case returns after trying all options

Exercises

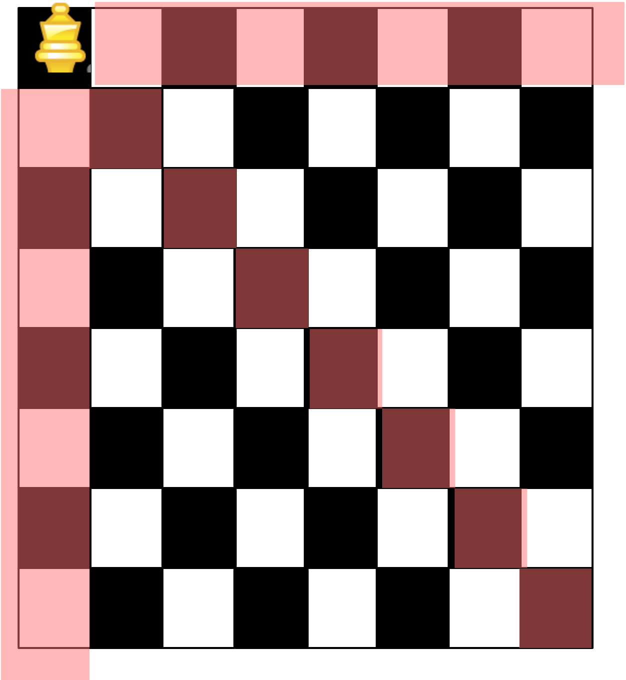
- bin_combos_str
- Zero_sum
- Prime_products_print
- Prime_products
- basen_combos
- all_letter_combos

Recursive Backtracking Search

- Recursion allows us to "easily" enumerate all **solutions/combinations** to some problem
- Backtracking algorithms are often used to solve **constraint satisfaction problems or optimization problems**
  - Find (**the best**) solutions/combinations that meet **some constraints**
- **Key property of backtracking search:**
  - Stop searching down a path at the first indication that constraints won't lead to a solution
- Many common and important problems can be solved with backtracking approaches
- Knapsack problem
  - You have a set of products with a given weight and value. Suppose you have a knapsack (suitcase) that can hold N pounds, which subset of objects can you pack that **maximizes the value**.
  - Example:
    - Knapsack can hold 35 pounds
    - Product A: 7 pounds, $12 ea. Product B: 10 pounds, $18 ea.
    - Product C: 4 pounds, $7 ea. Product D: 2.4 pounds, $4 ea.
- Other examples:
  - Map Coloring, Satisfiability, Sudoku, N-Queens

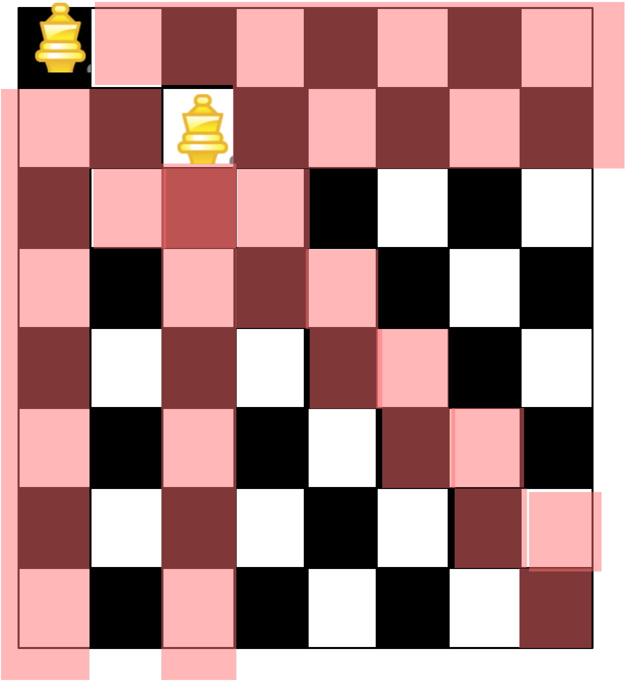
N-Queens Problem

- Problem: How to place N queens on an NxN chess board such that no queens may attack each other
- Fact: Queens can attack at any distance vertically, horizontally, or diagonally
- Observation: Different queen in each row and each column
- Backtrack search approach:
  - Place 1st queen in a viable option then, then try to place 2nd queen, etc.
  - If we reach a point where no queen can be placed in row i or we've exhausted all options in row i, then we return and change row i-1



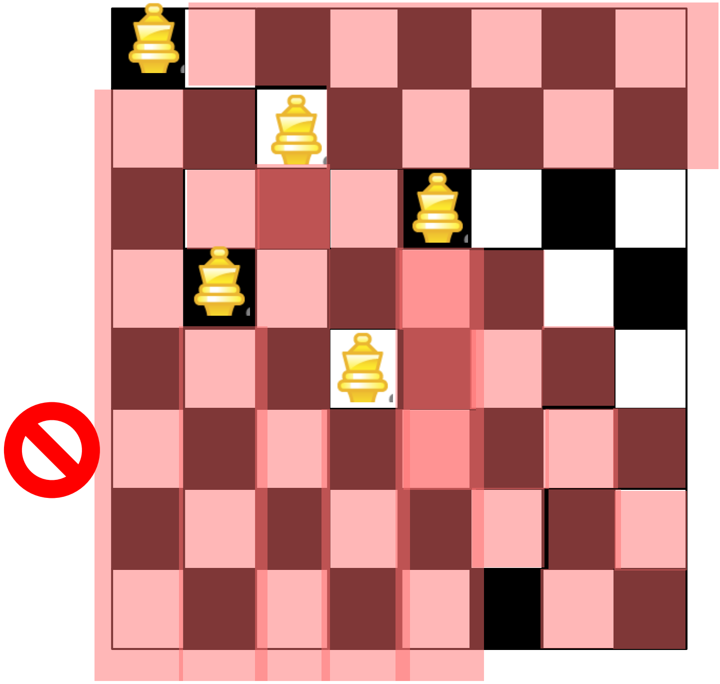
8x8 Example of N-Queens

- Now place 2nd queen



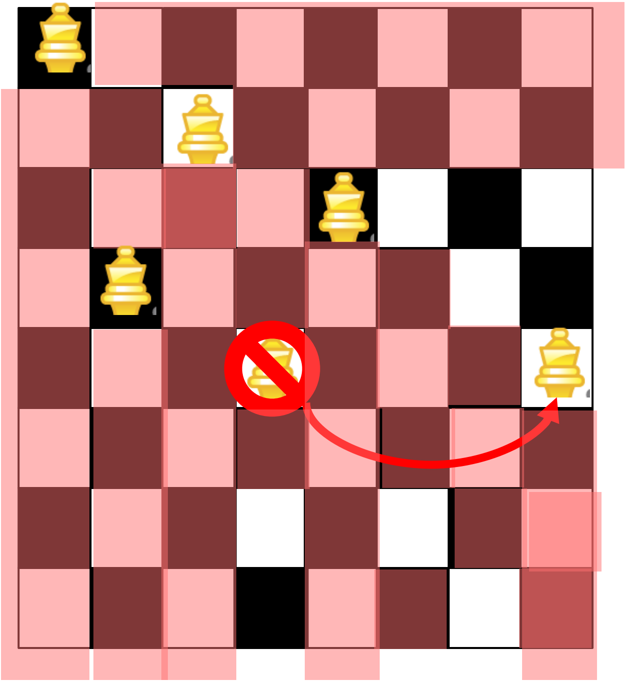
8x8 Example of N-Queens

- Now place others as viable
- After this configuration here, there are no locations in row 6 that are not under attack from the previous 5
- BACKTRACK!!!



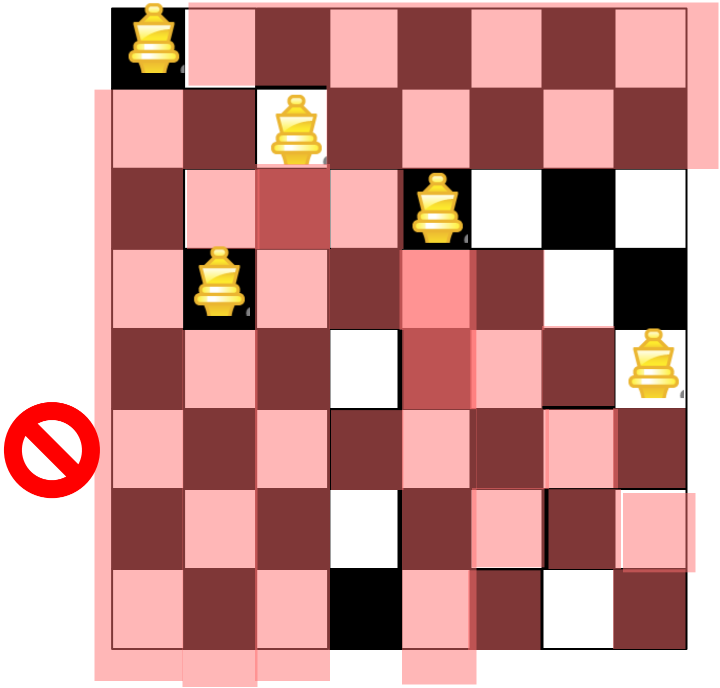
8x8 Example of N-Queens

- Now place others as viable
- After this configuration here, there are no locations in row 6 that is not under attack from the previous 5
- So go back to row 5 and switch assignment to next viable option and progress back to row 6



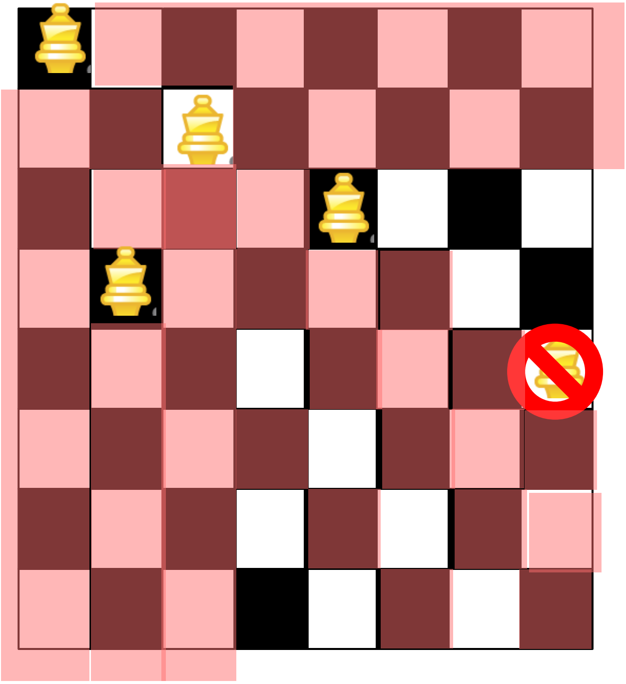
8x8 Example of N-Queens

- Now place others as viable
- After this configuration here, there are no locations in row 6 that is not under attack from the previous 5
- Now go back to row 5 and switch assignment to next viable option and progress back to row 6
- But still no location available so return back to row 5

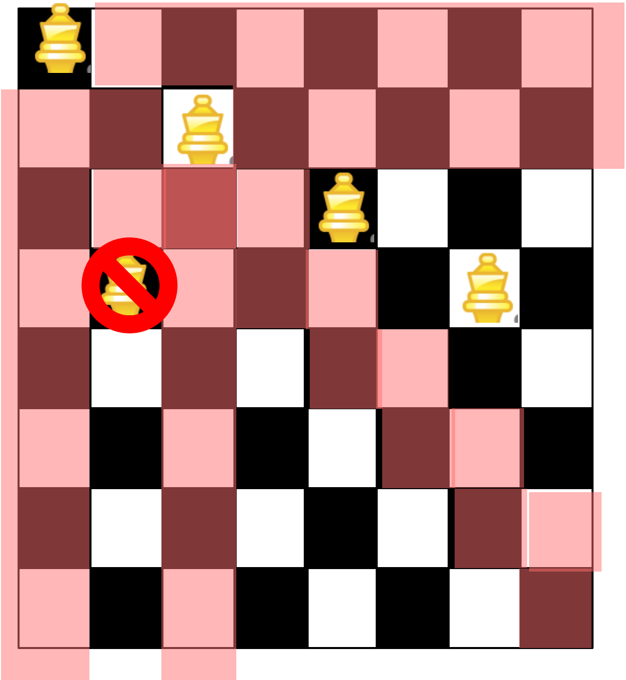


8x8 Example of N-Queens

- Now place others as viable
- After this configuration here, there are no locations in row 6 that is not under attack from the previous 5
- Now go back to row 5 and switch assignment to next viable option and progress back to row 6
- But still no location available so return back to row 5
- But now no more options for row 5 so return back to row 4
- BACKTRACK!!!!



8x8 Example of N-Queens



- Now place others as viable
- After this configuration here, there are no locations in row 6 that is not under attack from the previous 5
- Now go back to row 5 and switch assignment to next viable option and progress back to row 6
- But still no location available so return back to row 5
- But now no more options for row 5 so return back to row 4
- Move to another place in row 4 and restart row 5 exploration

8x8 Example of N-Queens

- Now place others as viable
- After this configuration here, there are no locations in row 6 that is not under attack from the previous 5
- Now go back to row 5 and switch assignment to next viable option and progress back to row 6
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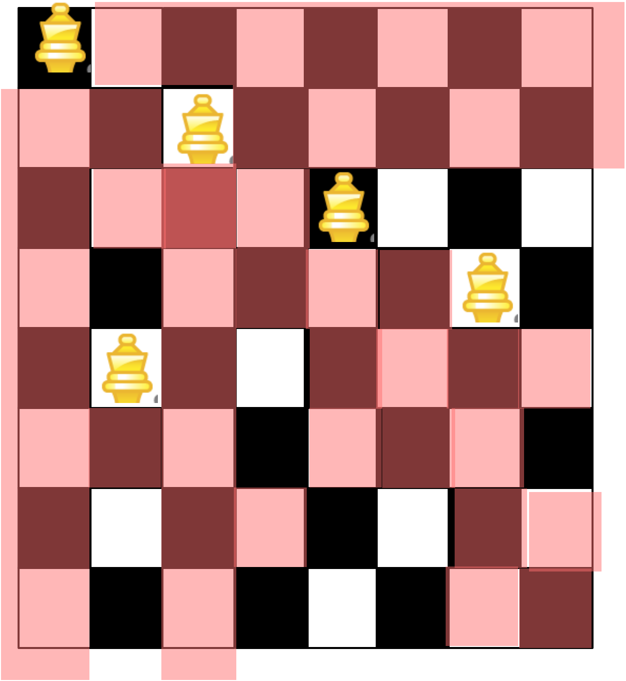
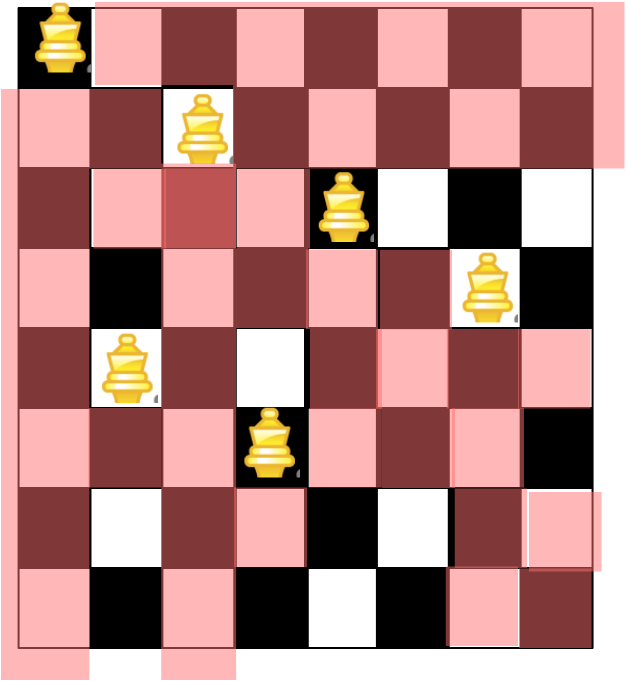


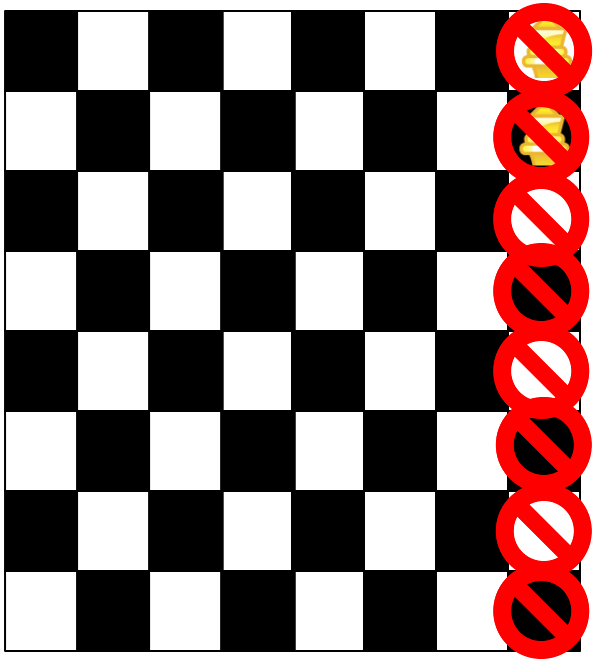
8x8 Example of N-Queens



- Now a viable option exists for row 6
- Keep going until you successfully place row 8 in which case you can return your solution
- What if no solution exists?

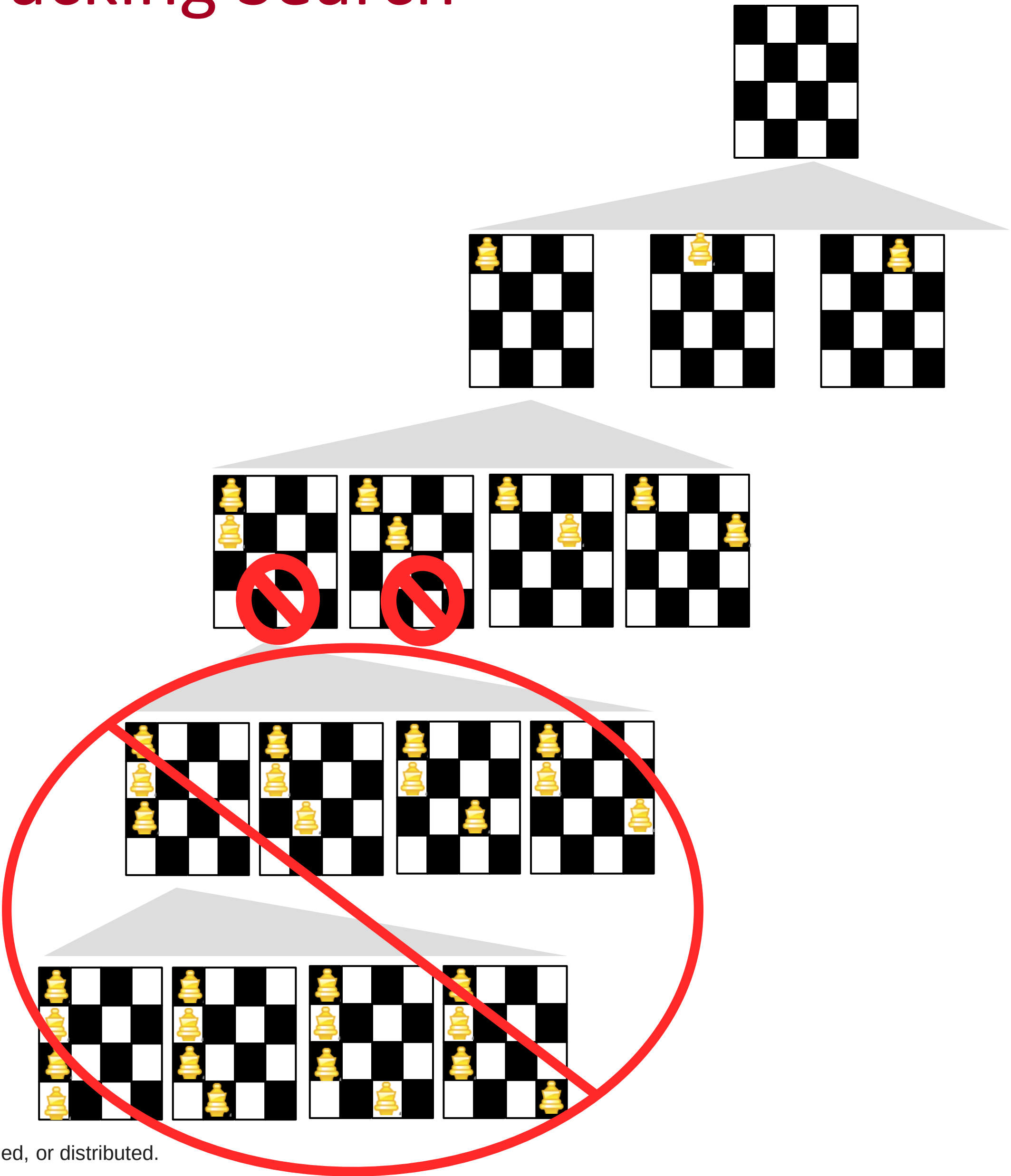
8x8 Example of N-Queens

- Now a viable option exists for row 6
- Keep going until you successfully place row 8 in which case you can return your solution
- What if no solution exists?
  - Row 1 queen would have exhausted all her options and still not find a solution



Backtracking Search

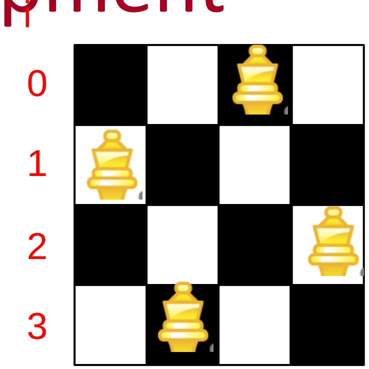
- Recursion can be used to generate all options
  - 'brute force' / test all options approach
  - Test for constraint satisfaction only at the bottom of the 'tree'
- But backtrack search attempts to 'prune' the search space
  - Rule out options at the partial assignment level



Brute force enumeration might test only when a complete assignment is made (i.e. all 4 queens on the board)

N-Queens Solution Development

- Let's develop the code
- 1 queen per row
  - Use an array where index represents the queen (and the row) and value is the column
- Start at row 0 and initiate the search [i.e. search(0)]
- Base case:
  - Rows range from 0 to n-1 so STOP when row == n
  - Means we found a solution
- Recursive case
  - Recursively try all column options for that queen
  - But haven't implemented check of viable configuration…



Index = Queen i in row i	**0**	**1**	**2**	**3**
q[i] = column of queen i	**2**	**0**	**3**	**1**

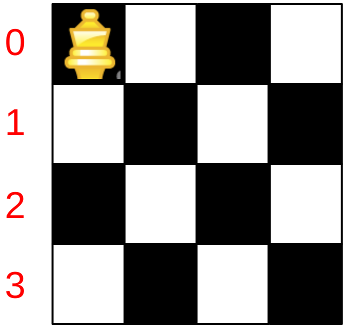
```
int *q;  // pointer to array storing
         // each queens location
int n;   // number of board / size

void search(int row)
{
  if(row == n)
    printSolution(); // solved!
  else {
   // remember q[row] is the column
   for(q[row]=0; q[row]<n; q[row]++){
     search(row+1);

   // alternatively
//   for(int col = 0; col < n; col++){
//      q[row] = col;
//      search(row+1);
//   }
}
```

N-Queens Solution Development

- To check whether it is safe to place a queen in a particular column, let's keep a "threat" 2-D array indicating the threat level at each square on the board
  - Threat level of 0 means SAFE
  - When we place a queen we'll update squares that are now under threat
  - Let's name the array 't'
- Dynamically allocating 2D arrays in C/C++ doesn't really work
  - Instead conceive of 2D array as an "array of arrays" which boils down to a pointer to a pointer

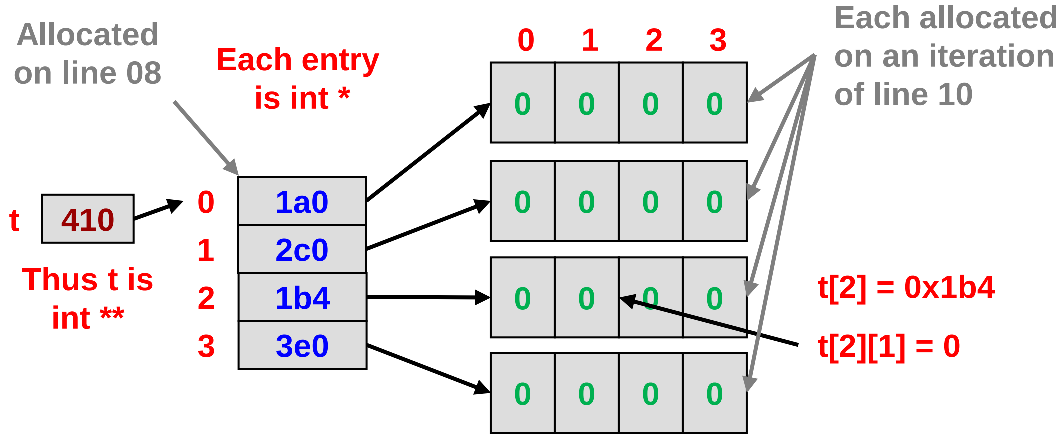


0	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1

Index = Queen i in row i

q[i] = column of queen i

0	1	2	3
0			



Allocated on line 08

Each entry is int *

Thus t is int **

Each allocated on an iteration of line 10

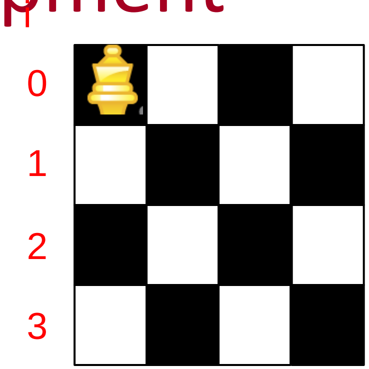
t[2] = 0x1b4

t[2][1] = 0

```
00  int *q;   // pointer to array storing
01            // each queens location
02  int n;    // number of board / size
03  int **t; // thread 2D array
04
05  int main()
06  {
07    q = new int[n];
08    t = new int*[n];
09    for(int i=0; i < n; i++){
10      t[i] = new int[n];
11      for(int j = 0; j < n; j++){
12        t[i][j] = 0;
13      }
14    }
15    search(0); // start search
16    // deallocate arrays
17    return 0;
18  }
```

N-Queens Solution Development

- After we place a queen in a location, let's check that it has no threats
- If it's safe then we update the threats (+1) due to this new queen placement
- Now recurse to next row
- If we return, it means the problem was either solved or more often, that no solution existed given our placement so we remove the threats (-1)
- Then we iterate to try the next location for this queen



Index = Queen i in row i

q[i] = column of queen i

0	1	2	3
0			

```
int *q;  // pointer to array storing
         // each queens location
int n;   // number of board / size
int **t; // n x n threat array
void search(int row)
{
  if(row == n)
    printSolution(); // solved!
  else {
   for(q[row]=0; q[row]<n; q[row]++){
     // check that col: q[row] is safe
     if(t[row][q[row]] == 0){
       // if safe place and continue
       addToThreats(row, q[row], 1);
       search(row+1);
       // if return, remove placement
       addToThreats(row, q[row], -1);
} } }
```

t	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

Safe to place queen in upper left

t	0	1	2	3
0	0	1	1	1
1	1	1	0	0
2	1	0	1	0
3	1	0	0	1

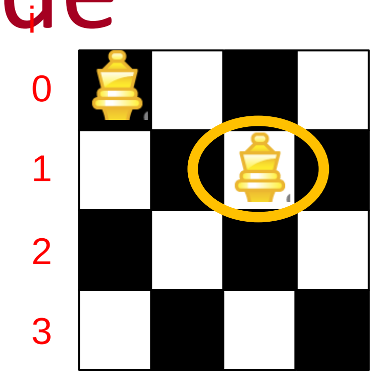
Now add threats

t	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

Upon return, remove threat and iterate to next option

addToThreats Code

- Observations
  - Already a queen in every higher row so addToThreats only needs to deal with positions lower on the board
    - Iterate row+1 to n-1
  - Enumerate all locations further down in the same column, left diagonal and right diagonal
  - Can use same code to add or remove a threat by passing in change
- Can't just use 2D array of booleans as a square might be under threat from two places and if we remove 1 piece we want to make sure we still maintain the threat



Index = Queen i in row i

q[i] = column of queen i

Index	0	1	2	3
q[i]	0			

```
void addToThreats(int row, int col, int change)
{
  for(int j = row+1; j < n; j++){
    // go down column
    t[j][col] += change;
    // go down right diagonal
    if( col+(j-row) < n )
       t[j][col+(j-row)] += change;
    // go down left diagonal
    if( col-(j-row) >= 0)
       t[j][col-(j-row)] += change;
  }
}
```

t	0	1	2	3
0	0	1	1	1
1	1	1	0	0
2	1	0	1	0
3	1	0	0	1

t	0	1	2	3
0	0	1	1	1
1	1	1	0	0
2	1	1	2	1
3	2	0	1	1

N-Queens Solution

```
int *q;  // queen location array
int n;   // number of board / size
int **t; // n x n threat array

int main()
{
  q = new int[n];
  t = new int*[n];
  for(int i=0; i < n; i++){
    t[i] = new int[n];
    for(int j = 0; j < n; j++){
      t[i][j] = 0;
    }
  }
  // do search
  if( ! search(0) )
     cout << "No sol!" << endl;
  // deallocate arrays
  return 0;
}
```

```
void addToThreats(int row, int col, int change)
{
  for(int j = row+1; j < n; j++){
    // go down column
    t[j][col] += change;
    // go down right diagonal
    if( col+(j-row) < n )
       t[j][col+(j-row)] += change;
    // go down left diagonal
    if( col-(j-row) >= 0)
       t[j][col-(j-row)] += change;
  }
}

bool search(int row)
{
  if(row == n){
    printSolution(); // solved!
    return true;
  }
  else {
   for(q[row]=0; q[row]<n; q[row]++){
     // check that col: q[row] is safe
     if(t[row][q[row]] == 0){
       // if safe place and continue
       addToThreats(row, q[row], 1);
       bool status = search(row+1);
       if(status) return true;
       // if return, remove placement
       addToThreats(row, q[row], -1);
     }
   }
   return false;
} }
```

General Backtrack Search Approach

- Select an item and set it to one of its options such that it meets current constraints
- Recursively try to set next item
- If you reach a point where all items are assigned and meet constraints, done…return through recursion stack with solution
- If no viable value for an item exists, backtrack to previous item and repeat from the top
- If viable options for the 1st item are exhausted, no solution exists
- Phrase:
  - Assign, recurse, unassign

General Outline of Backtracking Sudoku Solver

```
bool sudoku(int **grid, int r, int c)
{
  if( allSquaresComplete(grid) )
    return true;
  }
  // iterate through all options
  for(int i=1; i <= 9; i++){
    grid[r][c] = i;
    if( isValid(grid) ){
      bool status = sudoku(...);
      if(status) return true;
    }
  }
  return false;
}
```

Assume r,c is current square to set and grid is the 2D array of values

Runtime of All Combinations

- $T(n_r, n_c) =$ ________________
- $T(0, n_c) = 1$

```
int *q;  // pointer to array storing
         // each queens location
int n;   // number of board / size

void search(int row)
{
  if(row == n)
    printSolution(); // solved!
  else {
   // remember q[row] is the column
   for(q[row]=0; q[row]<n; q[row]++){
     search(row+1);
   }
}
```

SOLUTIONS

Recursion and DFS

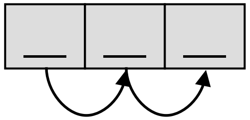
- Recursion forms a kind of Depth-First Search

Options

0
1

Generally: Recursion must perform the same code sequence for each item. Where we need variation, use 'if' statements.

N = length

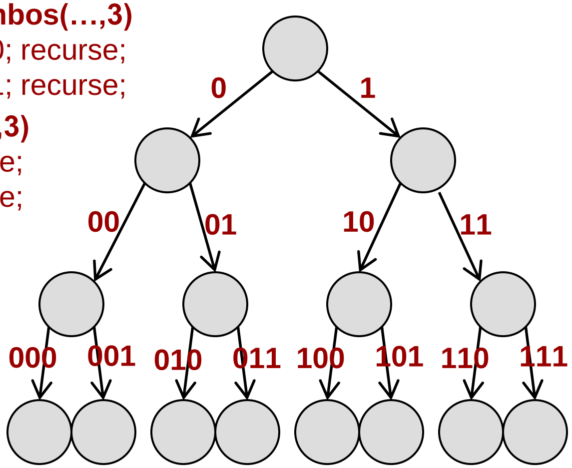


binCombos(…,3)
Set to 0; recurse;
Set to 1; recurse;

binCombos(…,3)
Set to 0; recurse;
Set to 1; recurse;

binCombos(…,3)
Set to 0; recurse;
Set to 1; recurse;

binCombos(…,3)
Base case



```
// user interface
void binCombos(int len)
{
  binCombos("", len);
}

// helper-function
void binCombos(string prefix,
               int len)
{
  if(prefix.length() == len )
    cout << prefix << endl;
  else {
    // recurse
    binCombos(prefix+"0" len);
    // recurse
    binCombos(prefix+"1", len);
  }
}
```