CSCI 104
Runtime Complexity

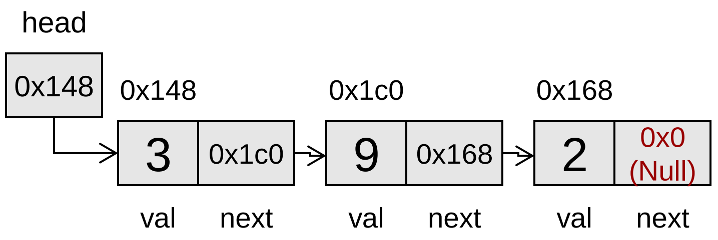
Mark Redekopp
David Kempe

Runtime

- It is hard to compare the run time of an algorithm on actual hardware
  - Time may vary based on speed of the HW, etc.
    - The same program may take 1 sec. on your laptop but 0.5 second on a high performance server
- If we want to compare 2 algorithms that perform the same task we could try to count operations (regardless of how fast the operation can execute on given hardware)…
  - But what is an operation?
  - How many operations is: i++ ?
  - i++ actually requires grabbing the value of i from memory and bringing it to the processor, then adding 1, then putting it back in memory. Should that be 3 operations or 1?
  - Its painful to count 'exact' numbers operations
- Big-O, Big-Ω, and Θ notation allows us to be more general (or "sloppy" as you may prefer)

Complexity Analysis

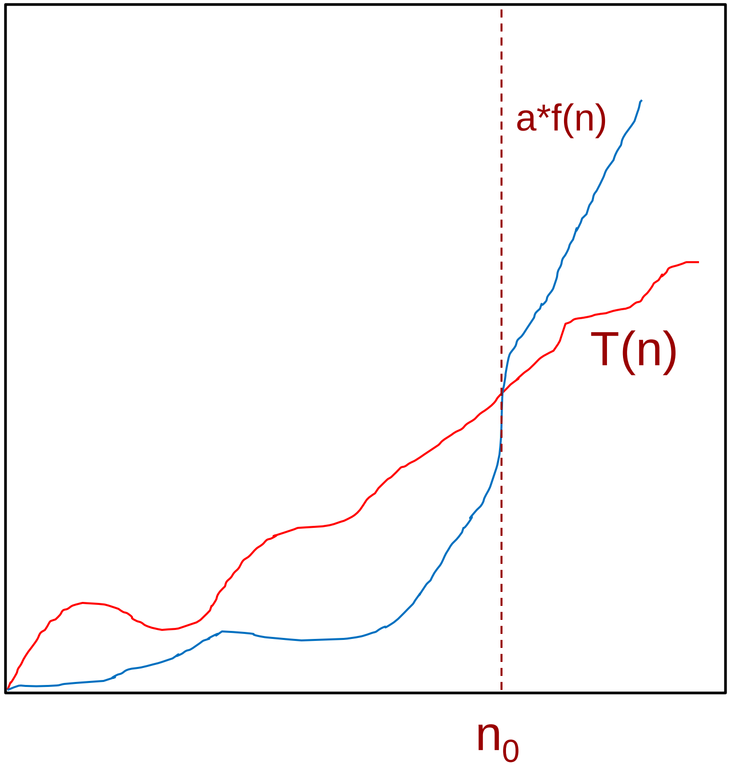
- To find upper or lower bounds on the complexity, we must consider the set of all possible inputs, I, of size, n
- Derive an expression, T(n), in terms of the input size, n, for the number of operations/steps that are required to solve the problem of a given input, i
  - Some algorithms depend on i and n
    - Find(3) in the list shown vs. Find(2)
  - Others just depend on n
    - Push_back / Append
- Which inputs though?
  - Best, worst, or "typical/average" case?
- We will always apply it to the "worst case"
  - That's usually what people care about



Note: Running time is not just based on an algorithm, BUT algorithm + input data

Big-O, Big-Ω

- T(n) is said to be O(f(n)) if…
  - T(n) < a*f(n) for n > n_0 (where a and n_0 are constants)
  - Essentially an upper-bound
  - We'll focus on big-O for the worst case
- T(n) is said to be Ω(f(n)) if…
  - T(n) > a*f(n) for n > n_0 (where a and n_0 are constants)
  - Essentially a lower-bound
- T(n) is said to be Θ(f(n)) if…
  - T(n) is both O(f(n)) AND Ω(f(n))



Worst Case and Big-Ω

- What's the lower bound on List::find(val)
  - Is it $\Omega(1)$ since we might find the given value on the first element?
  - Well it could be if we are finding a lower bound on the 'best case'
- Big-Ω does ***NOT*** have to be ***synonymous*** with 'best case'
  - Though many times it mistakenly is
- You can have:
  - Big-O for the best, average, worst cases
  - Big-Ω for the best, average, worst cases
  - Big-Θ for the best, average, worst cases

Worst Case and Big-Ω

- The key idea is an algorithm may perform differently for different input cases
  - Imagine an algorithm that processes an array of size n but depends on what data is in the array
- Big-O for the ***worst-case*** says ***ALL*** possible inputs are bound by O(f(n))
  - Every possible combination of data is at MOST bound by O(f(n))
- Big-Ω for the ***worst-case*** is attempting to establish a lower bound (at-least) for the worst case (the worst case is just one of the possible input scenarios)
  - If we look at the first data combination in the array and it takes n steps then we can say the algorithm is Ω(n).
  - Now we look at the next data combination in the array and the algorithm takes $n^{1.5}$. We can now say worst case is $\Omega(n^{1.5})$.
- To arrive at Ω(f(n)) for the ***worst-case*** requires you simply to find ***AN*** input case (i.e. the worst case) that requires ***at least*** f(n) steps

Deriving T(n)

- Derive an expression, T(n), in terms of the input size for the number of operations/steps that are required to solve a problem
- If is true => 4
- Else if is true => 5
- Worst case => T(n) = 5

```
#include <iostream>

using namespace std;


int main()
{

  int i = 0;             1

  x = 5;                 1

  if(i < x){             1
     x--;                1
  }
  else if(i > x){        1
     x += 2;             1
  }
  return 0;
}
```

Deriving T(n)

- Since loops repeat you have to take the sum of the steps that get executed over all iterations
- $T(n) =$
- $= \sum_{i=0}^{n-1} 5 = 5 * n$
- Or you can setup a relationship like:
- $T(n) = T(n-1) + 5$
- $= T(n-2) + 5 + 5$
- $= \sum_{i=0}^{n-1} 5 = 5 * n$
- $= \sum_{i=0}^{n-1} O(1) = O(n)$

```
#include <iostream>
using namespace std;

int main()
{

  for(int i=0; i < N; i++){
    x = 5;
    if(i < x){
       x--;
    }
    else if(i > x){
       x += 2;
    }
  }
  return 0;
}
```

Common Summations

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \theta(n^2)$
  - This is called the arithmetic series
- $\sum_{i=1}^{n} \theta(i^p) = \theta(n^{p+1})$
  - This is a general form of the arithmetic series
- $\sum_{i=1}^{n} c^i = \frac{c^{n+1}-1}{c-1} = \theta(c^n)$
  - This is called the geometric series
- $\sum_{i=1}^{n} \frac{1}{i} = \theta(\log n)$
  - This is called the harmonic series

Skills You Should Gain

- To solve these running time problems try to break the problem into 2 parts:
- FIRST, setup the expression (or recurrence relationship) for the number of operations
- SECOND, solve
  - Unwind the recurrence relationship
  - Develop a series summation
  - Solve the series summation

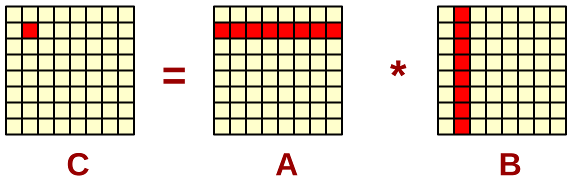
Loops

- Derive an expression, T(n), in terms of the input size for the number of operations/steps that are required to solve a problem
- $T(n) =$
- $= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \theta(1) = \sum_{i=0}^{n-1} \theta(n) = \Theta(n^2)$

```
#include <iostream>

using namespace std;
const int n = 256;
unsigned char image[n][n]
int main()
{
  for(int i=0; i < n; i++){
    for(int j=0; j < n; j++){
       image[i][j] = 0;
    }
  }
  return 0;
}
```

Matrix Multiply



Traditional Multiply

- Derive an expression, T(n), in terms of the input size for the number of operations/steps that are required to solve a problem
- $T(n) =$
- $= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \theta(1) = \theta(n^3)$

```
#include <iostream>
using namespace std;
const int n = 256;
int a[n][n], b[n][n], c[n][n];
int main()
{
  for(int i=0; i < n; i++){
    for(int j=0; j < n; j++){
      c[i][j] = 0;
      for(int k=0; k < n; k++){
        c[i][j] += a[i][k]*b[k][j];
      }
    }
  }
  return 0;
}
```

Sequential Loops

- Is this also n^3?
- No!
  - 3 for loops, but not nested
  - $O(n) + O(n) + O(n) = 3*O(n) = O(n)$

```
#include <iostream>

using namespace std;
const int n = 256;
unsigned char image[n][n]
int main()
{
  for(int i=0; i < n; i++){
    image[0][i] = 5;
  }
  for(int j=0; j < n; j++){
    image[1][j] = 5;
  }
  for(int k=0; k < n; k++){
    image[2][k] = 5;
  }
 return 0;
}
```

Counting Steps

- It may seem like you can just look for nested loops and then raise n to that power
  - 2 nested for loops => $O(n^2)$
- But be careful!!
- You have to count steps
  - Look at the update statement
  - Outer loop increments by 1 each time so it will iterate N times
  - Inner loop updates by dividing x in half each iteration?
  - After 1st iteration => x=n/2
  - After 2nd iteration => x=n/4
  - After 3rd iteration => x=n/8
  - Say k^{th} iteration is last => $x = n/2^k = 1$
  - Solve for k
  - $k = \log_2(n)$ iterations
  - O(n*log(n))

```
#include <iostream>
using namespace std;
const int n = 256;

int main()
{
  for(int i=0; i < n; i++){
    int y=0;
    for(int x=n; x != 1; x=x/2){
        y++;
    }
    cout << y << endl;
  }
  return 0;
}
```

Analyze This

- Count the steps of this example?

```
#include <iostream>
using namespace std;
const int n = 256;
int a[n];
int main()
{
  for(int i=0; i < n; i++){
    a[i] = 0;
    for(int j=0; j < i; j++){
        a[i] += j;
    }
  }
  return 0;
}
```

- $T(n) = T(n-1) + n-1$
- $0 + 1 + \ldots + n-2 + n-1$
- $(n-1)*n/2$

Analyze This

- Count the steps of this example?

```
for (int i = 0; i <= log2(n); i ++)
   for (int j=0; j < (int) pow(2,i); j++)
      cout << j;
```

- $\sum_{i=0}^{\lg(n)} \sum_{j=0}^{2^i} 1$
- $=\sum_{i=0}^{\lg(n)} 2^i$
- Use the geometric sum eqn.
- $=\sum_{i=0}^{n-1} a^i = \frac{1-a^n}{1-a}$
- So our answer is…
- $\frac{1-2^{\lg(n)+1}}{1-2} = \frac{1-2*n}{-1} = O(n)$

Another Example

- Count steps here…
  - Think about how many times if statement will evaluate true

```
for (int i = 0; i < n; i++)
{
   cout << "i: ";
   int m = sqrt(n);
   if( i % m == 0){
     for (int j=0; j < n; j++)
       cout << j << " ";
   }
   cout << endl;
}
```

- $T(n) = \sum_{i=0}^{n-1}(\theta(1) + O(n))$
- $T(n) =$

Another Example

- Count steps here…
  - Think about how many times if statement will evaluate true

```
for (int i = 0; i < n; i++)
{
   cout << "i: ";
   int m = sqrt(n);
   if( i % m == 0){
     for (int j=0; j < n; j++)
       cout << j << " ";
   }
   cout << endl;
}
```

- $T(n) = \sum_{i=0}^{n-1}(\theta(1) + O(n))$
- $T(n) = \sum_{i=0}^{n-1}\theta(1) + \sum_{k=1}^{\sqrt{n}}\sum_{j=1}^{n}\theta(1)$
- $T(n) = \theta(n) + \sum_{k=1}^{\sqrt{n}}\theta(n)$
- $T(n) = \theta(n) + \theta(n \cdot \sqrt{n})$
- $T(n) = \theta\left(n^{3/2}\right)$

What about Recursion

- Assume N items in the linked list
- $T(n) = 1 + T(n-1)$
- $= 1 + 1 + T(n-2)$
- $= 1 + 1 + 1 + T(n-3)$
- $= n = O(n)$

```
void print(Item* head)
{
   if(head==NULL) return;
   else {
     cout << head->val << endl;
     print(head->next);
   }
}
```

Binary Search

- Assume N items in the data array
- T(n) =
  - O(1) if base case
  - O(1) + T(n/2)
- = 1 + T(n/2)
- = 1 + 1 + T(n/4)
- = k + $T(n/2^k)$
- Stop when $2^k = n$
  - Implies $\log_2(n)$ recursions
- $O(\log_2(n))$

```
int bsearch(int data[],
            int start, int end,
            int target)
{
  if(end >= start)
    return -1;
  int mid = (start+end)/2;
  if(target == data[mid])
    return mid;
  else if(target < data[mid])
    return bsearch(data, start, mid,
                   target);
  else
    return bsearch(data, mid, end,
                   target);
}
```

AMORTIZED RUNTIME

Example

- You love going to Disneyland. You purchase an annual pass for $240. You visit Disneyland once a month for a year. Each time you go you spend $20 on food, etc.
  - What is the cost of a visit?
- Your annual pass cost is spread or "**amortized**" (or averaged) over the duration of its usefulness
- Often times an operation on a data structure will have similar "irregular" costs that we can then amortize over future calls

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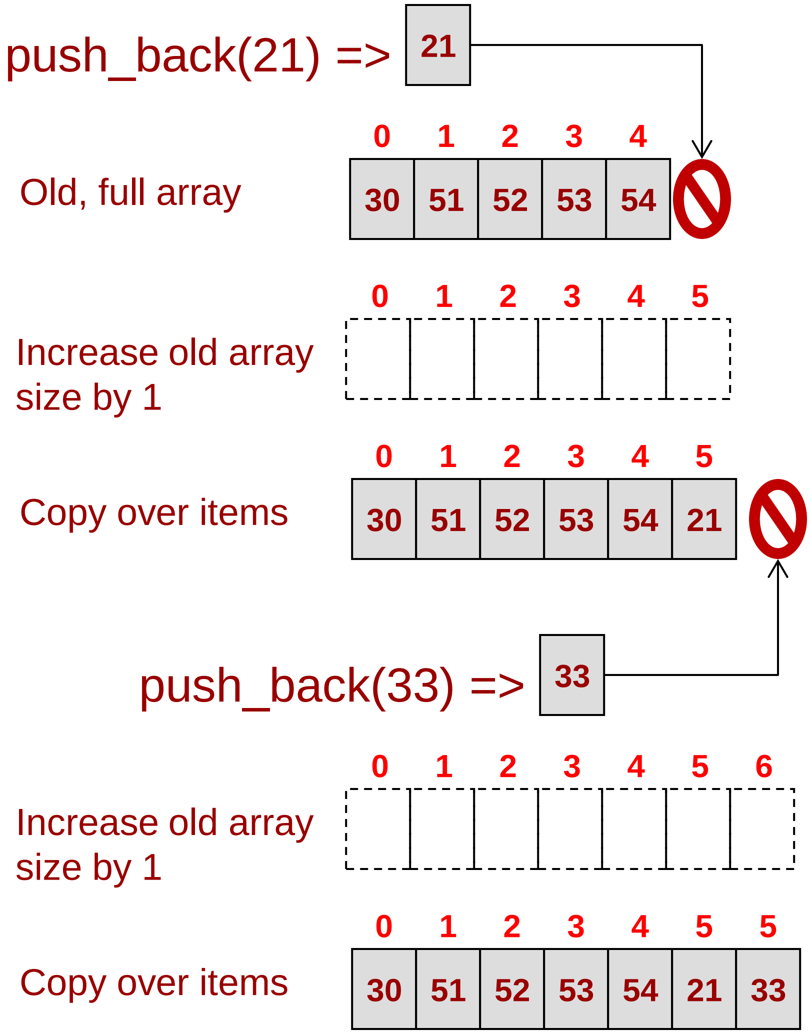
Amortized Array Resize Run-time

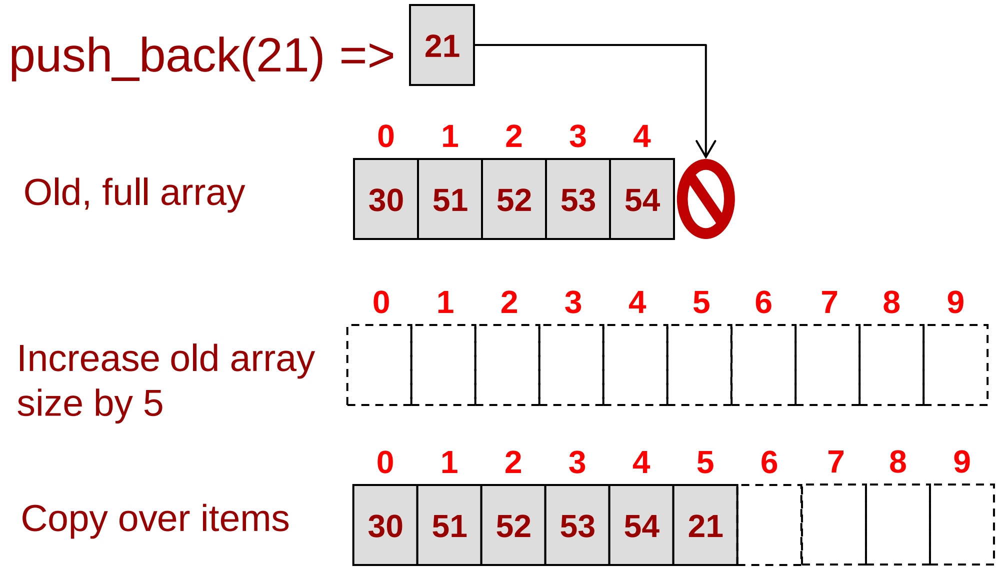
- What is the run-time of insert or push_back:
  - If we have to resize?
  - O(n)
  - If we don't have to resize?
  - O(1)
- Now compute the total cost of a series of insertions using resize by 1 at a time
- Each insert now costs O(n)… not good



Resize by 1 strategy

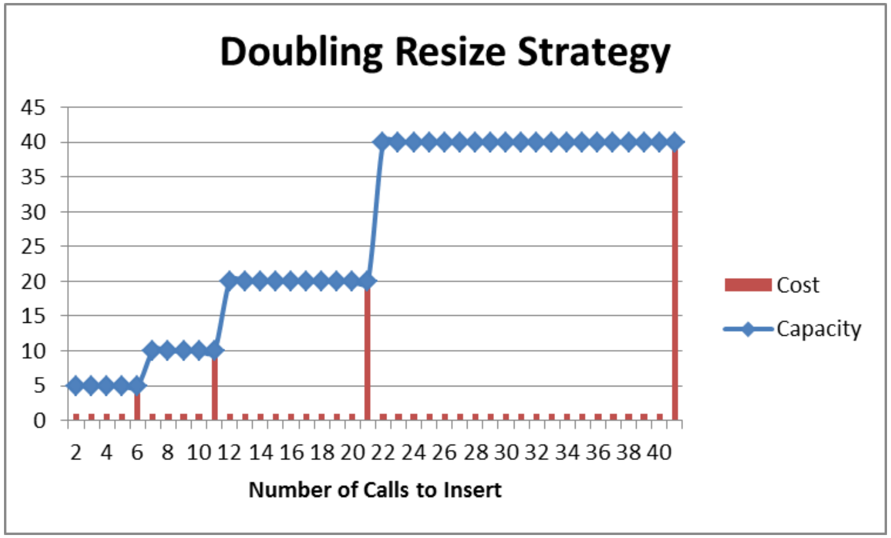
Amortized Array Resize Run-time

- What if we resize by adding 5 new locations each time
- Start analyzing when the list is full…
  - 1 call to insert will cost: 5
  - What can I guarantee about the next 4 calls to insert?
    - They will cost 1 each because I have room
  - After those 4 calls the next insert will cost: 10
  - Then 4 more at cost=1
- If the list is size n and full
  - Next insert cost = n
  - 4 inserts after than = 1 each
  - Cost for 5 inserts = n+5
  - Runtime = cost / insert = (n+5)/5 = O(n)



Consider a Doubling Size Strategy

- Start when the list is full and at size n
- Next insertion will cost?
  - O(n+1)
- How many future insertions will be guaranteed to be cost = 1?
  - n-1 insertions
  - At a cost of 1 each, I get n-1 total cost
- So for the n insertions my total cost was
  - n+1 + n-1 = 2*n
- Amortized runtime is then:
  - Cost / insertions
  - O(2*n / n) = O(2)
    = O(1) = constant!!!



Another Example

- Let's say you are writing an algorithm to take a n-bit binary combination (3-bit and 4-bit combinations are to the right) and produce the next binary combination
- Assume all the cost in the algorithm is spent changing a bit (define that as 1 unit of work)
- I could give you any combination, what is the worst case run-time? Best-case?
  - O(n) => 011 to 100
  - O(1) => 000 to 001

3-bit Binary
000
001
010
011
100
101
110
111

4-bit Binary
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111

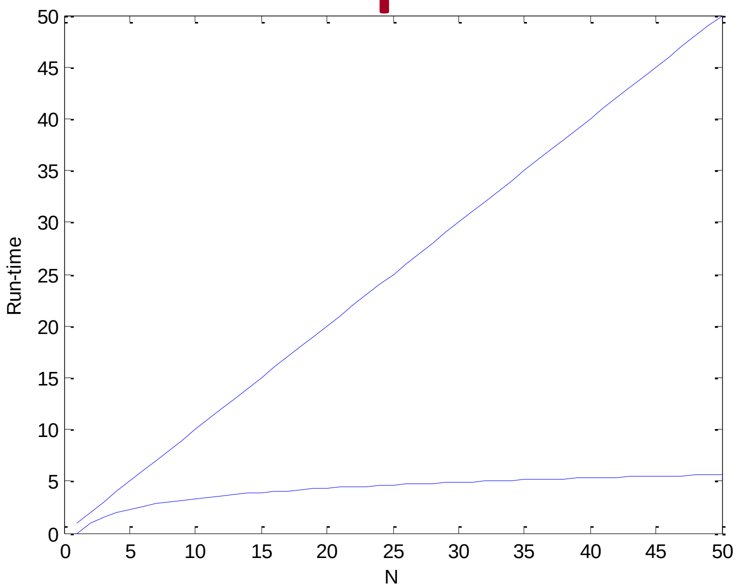
Another Example

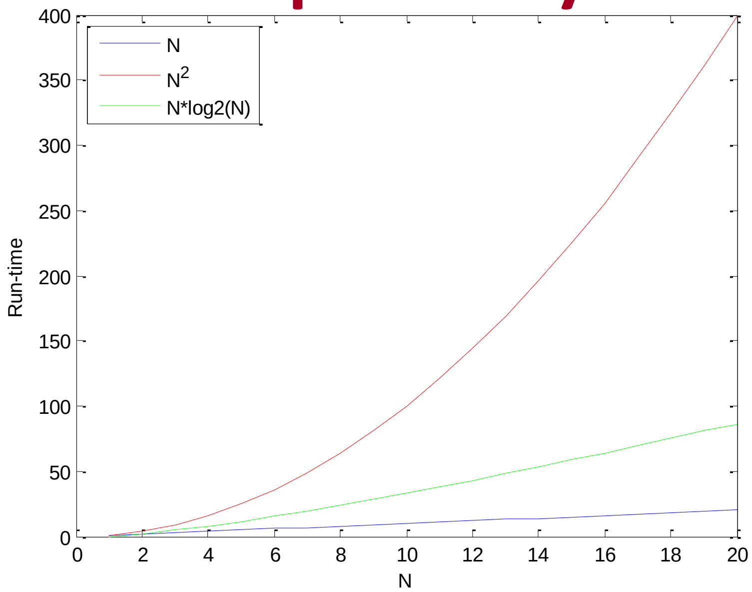
- Now let's consider the program that generates all the combinations sequentially (in order)
  - Starting at 000 => 001 : cost = 1
  - Starting at 001 => 010 : cost = 2
  - Starting at 010 => 011 : cost = 1
  - Starting at 011 => 100 : cost = 3
  - Starting at 100 => 101 : cost = 1
  - Starting at 101 => 110 : cost = 2
  - Starting at 101 => 111 : cost = 1
  - Starting at 111 => 000 : cost = 3
  - Total = 14 / 8 calls = 1.75
- Repeat for the 4-bit
  - 1 + 2 + 1 + 3 + 1 + 2 + 1 + 4 + …
  - Total = 30 / 16 = 1.875
- As n gets larger…Amortized cost per call = 2

3-bit Binary
000
001
010
011
100
101
110
111

4-bit Binary
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111

Importance of Complexity





N	O(1)	O(log_2n)	O(n)	O(n*log_2n)	O(n^2)	O(2^n)
2	1	1	2	2	4	4
20	1	4.3	20	86.4	400	1,048,576
200	1	7.6	200	1,528.8	40,000	1.60694E+60
2000	1	11.0	2000	21,931.6	4,000,000	#NUM!