Transient Performance Bounds for Adaptive Control

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***Abstract*— In this paper, we develop a methodology to understand and analyze transient performance in adaptive switching control systems that employ hysteresis type switching algorithms. In particular, we quantify the transient performance in terms of percent overshoot bound of the cost function (performance metric). The overshoot essentially measures the amount by which the achieved cost exceeds the robust cost, which is defined as the supremum of minimum controller cost over all possible input-output data and time. Furthermore, we highlight the parameters that can reduce this bound, which in turn improves the transient performance of the system. The efficacy of the proposed ideas are validated using simulation examples.**

I. INTRODUCTION

Ensuring satisfactory transient performance of adaptive control systems is a challenging task and is crucial, especially for safety-critical systems. Several attempts have been made to analyze and quantify transient performance of adaptive control systems in the past (see for example, [1], [2]). The most recent advancement was proposed in [3], which advocates the use of Closed loop Reference Models (CRMs) for transient performance improvement when compared to the traditional Open-loop Reference model (ORM) adaptive systems. The perspective adopted in these works follow the Model Reference Adaptive Control framework with an underlying state space model of the plant.

In this paper, we adopt an input-output approach to system analysis and develop a methodology to analyze the transient performance of the class of direct adaptive switching control systems wherein an adaptive algorithm selects an active controller by minimizing a cost function which depends on data and the past history of active controllers. One such class of adaptive systems is the one that employs purely data-driven gain-related cost functions [4]–[10] and the celebrated Morse-Mayne-Goodwin hysteresis type controller switching algorithms [11], [12]. By a data-driven cost function, we mean a performance metric that is a causal function of raw plant input-output data, uninterpreted by plant/noise models or other prior assumptions like 'minimum phase property of plant' or 'tunability', which helps prevent model-mismatch problems that can sometimes cause the adaptive algorithms to converge to destabilizing controllers [4], [13]. Of course plant assumptions can, and normally do, play an essential role in determining promising candidate controllers from which the adaptive switching control algorithm selects a suitable controller.

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For the class of adaptive control systems considered in this paper, convergence is well understood as shown in [4]. It was shown in [4] that when the cost functions have the 'cost-detectability' property and the adaptive control problem is 'feasible', the adaptive algorithm is convergent and stabilizing. However, even with feasibility and a cost-detectable cost function, an adaptive control system could still sometimes have unacceptably large transients as shown by an example in [14], illustrating that the transient response is not very well understood for this class of systems.

Consequently, we develop a theory to understand and quantify the transient performance in terms of a bound on the percent overshoot of an ℓ_2 gain related cost function. The overshoot here measures the amount by which the achieved cost exceeds the robust cost (defined later in the paper) of the adaptive system. Our goal is to achieve the smallest overshoot, therefore, we highlight the parameters that can reduce the bound on the percent overshoot, which in turn alleviates the possibility of bad transients in the system. Simulation examples validating the proposed ideas are provided. Although the theoretical proofs that follow in this paper assume discrete time systems, it is possible to obtain similar results for continuous time systems as well. Some of the techniques used to develop the theory in this paper have been inspired from [15], [16], wherein a different but, theoretically closely related problem of adaptive control resetting was analyzed.

The contents of this paper are organized as follows. Section II introduces the relevant notation. The structure of the switching adaptive control system, controller realization, preliminary concepts of a model free direct adaptive control scheme, hysteresis switching algorithm and the transient performance problem formulation are discussed in Section III. Key theoretical results that derive a bound on the percent overshoot of the cost function are presented in Section IV. Approaches to reduce this overshoot bound along with simulation examples are given in Section V. Conclusions are provided in VI.

II. NOTATION

Let $\mathbb{R}_+, \mathbb{Z}_+$ denote the set of non-negative real numbers and non-negative integers, respectively. Let $x : \mathbb{Z}_+ \to \mathbb{R}^n$, then the ℓ_p-norm of x is defined as

$$\|x\|_{p,[t_0,\,t]} = \begin{cases} \sqrt[p]{\Sigma_{\tau=t_0}^{t} \Sigma_{i=1}^{n} |x_i(\tau)|^p}, & \text{if } p \in [1, \infty) \\ \max_{\tau\in[t_0,\,t]} \max_{i\in[1,\,n]} |x_i(\tau)|, & \text{if } p = \infty \end{cases} \quad (1)$$

where $t \geq t_0$ and x_i denotes the i-th component of x. If $t_0 = 0$ and $\|x\|_{p,[0,\,t]} < \infty$ for $t \in [0, \infty)$, then $x \in \ell_{pe}^n$ and if $\lim_{t\to\infty} \|x\|_{p,[0,\,t]} < \infty$, then $x \in \ell_p^n$. The $\ell_{\lambda p}$-norm of x is defined as

$$\|x\|_{\lambda p,[t_0,\,t]} = \begin{cases} \sqrt[p]{\sum_{\tau=t_0}^{t} \lambda^{p(t-\tau)} \{\sum_{i=1}^{n} |x_i(\tau)|^p\}}, & \text{if } p \in [1,\infty) \\ \max_{\tau\in[t_0,\,t]} \lambda^{(t-\tau)} \{\max_{i\in[1,\,n]} |x_i(\tau)|\}, & \text{if } p = \infty \end{cases} \tag{2}$$

where λ is called the fading memory parameter and $\lambda \in (0, 1]$. The $\ell_{\lambda p}$-norm satisfies the following property,

$$\|x\|_{\lambda p,[0,\,t]} \le \|x\|_{\lambda p,[0,\,\tau-1]} \lambda^{t-\tau+1} + \|x\|_{\lambda p,[\tau,\,t]}, \quad \forall \tau \le t \tag{3}$$

It can be verified that (3) is an immediate consequence of (2) and triangle inequality property of norms.

The definition of λ-stability, λ-unfalsification and $\ell_{\lambda p}$ gain of a system are given below.

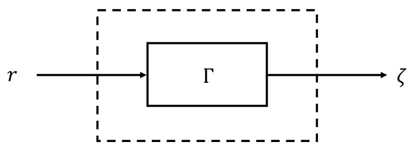

Fig. 1: A general system Γ

Definition 1. (λ-*stability*): Given a $\lambda \in (0, 1]$, the system $\Gamma : \ell_{pe}^n \to \ell_{pe}^m$, as shown in Fig. 1 with input r of size n and output ζ of size m, is λ-*stable* if for every input $r \in \ell_{pe}^n$, there exist constants $\alpha \ge 0$, $\beta \ge 0$ such that for all $t \ge 0$,

$$\|\zeta\|_{\lambda p,[0,\,t]} \le \beta \|r\|_{\lambda p,[0,\,t]} + \alpha. \tag{4}$$

When $\lambda = 1$, we simply say Γ is *stable*. □

Definition 2. (λ-*unfalsification*): Given a $\lambda \in (0, 1]$, a particular input r and output ζ, the stability of system Γ is said to be λ-*unfalsified* by the data pair (r, ζ) if there exist constants $\alpha \ge 0$, $\beta \ge 0$ such that, for all $t \ge 0$, we have

$$\|\zeta\|_{\lambda p,[0,\,t]} \le \beta \|r\|_{\lambda p,[0,\,t]} + \alpha \tag{5}$$

where $r \in \ell_{pe}^n$. Otherwise, the stability of Γ is λ-*falsified* by (r, ζ). When $\lambda = 1$, we simply say that stability is *unfalsified* if (5) holds, or *falsified* if (5) does not hold. □

Remark 1: The notion of λ-stability is equivalent to exponential stability of systems. Suppose Γ as shown in Fig. 1 is a finite dimensional Linear Time Invariant (LTI) system with transfer function $\Gamma(z)$, then Γ is λ-stable iff every pole of $\Gamma(z)$ has magnitude strictly less than λ. Also, λ-*stability* of $\Gamma(z)$ is equivalent to *stability* of $\Gamma(\lambda z)$.

Definition 3. ($\ell_{\lambda p}$ *gain*): The $\ell_{\lambda p}$ *gain* of the system Γ as shown in Fig. 1 is defined as

$$\|\Gamma\|_{\lambda p} = \sup_{\|r\|_{\lambda p,[0,\,t]}\neq 0,\; t\ge 0} \frac{\|\zeta\|_{\lambda p,[0,\,t]}}{\|r\|_{\lambda p,[0,\,t]}} \tag{6}$$

where $\zeta = \Gamma r$, $p \in [1,\infty]$ and $\lambda \in (0, 1]$. When $\lambda = 1$, its simply the ℓ_p gain of Γ. □

It follows from the above definition that the $\ell_{\lambda p}$ gain of a system Γ, satisfies the following property,

$$\|\Gamma r\|_{\lambda p,[0,\,t]} \le \|\Gamma\|_{\lambda p} \|r\|_{\lambda p,[0,\,t]} \tag{7}$$

for all $t \ge 0$.

III. PROBLEM FORMULATION

A. Switching Adaptive Control System

We consider a switching adaptive control system, $\Gamma : \ell_{pe}^n \to \ell_{pe}^m$, comprising of a high level controller called the supervisor and a finite candidate controller set $\mathbb{K} = \{K_1, K_2, ..., K_M\}$ in feedback with an uncertain plant P as shown in Fig. 2. The input to the system Γ is $r \in \ell_{pe}^n$ which denotes the reference signal and the output is $\zeta = [u\ y]^T \in \ell_{pe}^m$, where u and y denote the control signal and plant output signal respectively. The set $\mathbb{D} = \text{Graph}\{P\} := \{\zeta = (u, y) | y = Pu\}$. The supervisor along with the candidate controller set forms an adaptive switching controller. The supervisor orchestrates the switching of controllers and orders them using a cost function denoted as $V(K, \zeta, t)$, which is defined as a causal-in-time mapping $V : \mathbb{K} \times \mathbb{D} \times \mathbb{Z}_+ \longrightarrow \mathbb{R}_+ \cup \infty$. Given a specific ζ, we may for brevity denote $V_i = V(K_i, \zeta, t)$. The active controller at time t is denoted as $\hat{K}(t)$ and the closed loop switched system is denoted as $\Gamma(\hat{K}, P)$. The controller switching signal is denoted as σ, therefore $\sigma : \mathbb{Z}_+ \to \{1, 2, \ldots M\}$. The controller switching instants are denoted as $\{t_k\}_{k\in\mathbb{Z}_+}$ and the k-th switching interval is denoted as $T_k = \{t_k, t_k+1, \ldots, t_{k+1}-1\}$, over which σ remains constant. The total number of controller switches is denoted as n_s.

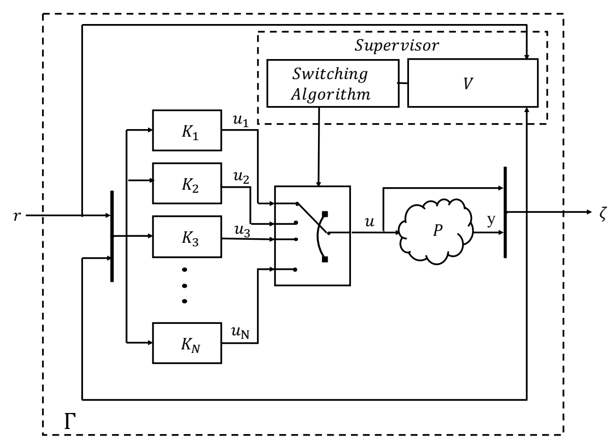

Fig. 2: A switching adaptive control system

B. Controller Realization

Let the ordered pair (N_i, D_i) denote the left Matrix Fraction Description (MFD) of a controller $K_i \in \mathbb{K}$. Therefore, the following holds,

$$K_i = D_i^{-1} N_i \tag{8}$$

where

$$N_i = [N_i^r \quad N_i^y];$$

factors N_i and D_i are stable, causal and D_i has a causal inverse [17]. The controller K_i can be nonlinear too, in which case it can be factored in terms of nonlinear incrementally stable factors [17], [18]. In this paper, the realization of the controller given by (8) is as shown in Fig. 3.

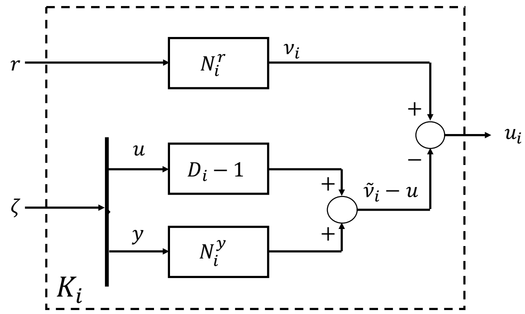

Fig. 3: Controller realization

With the above controller realization in place, the structure of the switching adaptive system is as shown in Fig. (4), Therefore, the control signal u is given by,

$$u = v_\sigma - (D_\sigma - 1)u - N_\sigma^y y \tag{9}$$

C. Model free Adaptive Control

Concepts pertaining to a plant model free direct adaptive control method [4] that are relevant to the current paper are provided in this section.

Definition 4. (*Fictitious/virtual Reference Signal (FRS)*): Given the plant input-output data $\zeta = [u \;\; y]^T$ from an experiment and a nonactive controller K_i having a stable matrix fraction description $K_i = D_i^{-1}[N_i^r \quad N_i^y]$, the *fictitious reference signal* $\tilde{v}_i$ is defined as a signal that would have reproduced exactly the measured data ζ, had the unswitched feedback controller law, $u_i = v_i - (D_i - 1)u - N_i^y y$ where $v_i = N_i^r r$, been in the loop during the entire time the data was collected. Therefore,

$$\tilde{v}_i = \Sigma_{\zeta,i}\zeta \tag{10}$$

where

$$\Sigma_{\zeta,i} = [D_i \quad N_i^y]. \tag{11}$$

□

Remark 2: When $N_i^r = N_i^y = I$ and $K_i = D_i^{-1}$, then $\tilde{v}_i$ equals the conventional fictitious reference signal $\tilde{r}_i$ in [19]. The use of the more general $\tilde{v}_i$ from [17] in place of the $\tilde{r}_i$ of [19] allows for the use of non-minimum phase controllers and the non-uniqueness of the stable matrix fraction description (N_i, D_i) provides additional flexibility that can be used to incorporate frequency-dependent weights in cost functions as well.

Definition 5. (*λ-cost-detectability*): Consider the switching adaptive control system Γ as shown in Fig. 2. A cost function and controller pair (V_i, K_i) is said to be λ *cost- detectable* if the following two statements are equivalent:
(1) $V_i \in \ell_\infty$.
(2) Stability of $\Gamma(K_i, P)$ is λ-unfalsified by the input-output data (r, ζ). □

Definition 6. (*Robust cost*): Given a particular plant P and controller K_i, the *robust cost* of controller K_i is denoted as $V_{\text{rsp}}(K_i)$ and is defined as $V_{\text{rsp}}(K_i) = \sup_{\zeta \in \mathbb{D}, t \in \mathbb{Z}_+} V_i$. A robust optimal controller is the one that minimizes the robust cost over all controllers in the candidate controller set. Let $\Pi_{\text{rsp}} = \min_{K_i \in \mathbb{K}} V_{\text{rsp}}(K_i)$ denote the robust cost of the system. □

Definition 7. (*λ-feasibility*): The adaptive stabilization problem is considered to be *λ-feasible* if there exists at least one controller in the candidate controller set $\mathbb{K}$ such that the closed loop system is λ-stable. In other words, the problem is λ-feasible if Π_{rsp} is finite when the controller cost function, V_i reflects λ-stability. □

D. Hysteresis Switching Algorithm

Hysteresis switching algorithm [11], [12] is one of the many switching algorithms used in adaptive switching control literature. The steps of the algorithm are provided below, *Hysteresis Algorithm A*1

1) Initialize: Let $t = 0$; choose $h > 0$: Let $\hat{K}(t) = K_0$; $K_0 \in \mathbb{K}$ be the first controller in the loop.
2) $t = t + 1$,
   If $V(\hat{K}(t-1), \zeta, t) \geq \min_{K_i \in \mathbb{K}} V(K_i, \zeta, t) + h$,
   then $\hat{K}(t) = \arg\min_{K_i \in \mathbb{K}} V(K_i, \zeta, t)$
   Else $\hat{K}(t) = \hat{K}(t-1)$
3) Go to Step 2. □

The quantity h is called the hysteresis constant and is positive. The total number of controller switches for hysteresis switching algorithm is bounded above by,

$$n_s = \left\lceil M \frac{\Pi_{\text{rsp}}}{h} \right\rceil \tag{12}$$

When the adaptive control problem is λ-feasible, n_s is finite.

Definition 8. (*Controller cost function*): In this paper, the following λ-cost-detectable *controller cost function* is used by the supervisor to orchestrate the switching,

$$V_i(t) = \max_{\tau \in [0,t]} \frac{\|\zeta\|_{\lambda p,[0,\tau]}}{\alpha + \|\tilde{v}_i\|_{\lambda p,[0,\tau]}} \tag{13}$$

where $V_i(t) = V(K_i, \zeta, t)$ and $\alpha \geq 0$. □

Definition 9. (*Achieved cost*): The *achieved cost* of the adaptive switching system is the actual reference (v_σ) to data induced gain such as the following,

$$V_\sigma(t) = \frac{\|\zeta\|_{\lambda p,[0,t]}}{\alpha + \|v_\sigma\|_{\lambda p,[0,t]}} \tag{14}$$

□

E. Transient Performance Analysis

When λ-feasibility holds, hysteresis switching algorithm along with a λ-cost-detectable cost function ensures convergence and λ-stability of the switching adaptive control system. However, in some cases, a destabilizing controller may be inserted repeatedly in the loop before finally stabilizing the system. This results in excessively large transient control and plant output signals. In this paper, we calculate a bound on the percent overshoot of the achieved cost. The percent overshoot is defined below.

Definition 10. (*Percent Overshoot*): *Percent Overshoot* denoted as *%OS* is defined as the maximum value of the

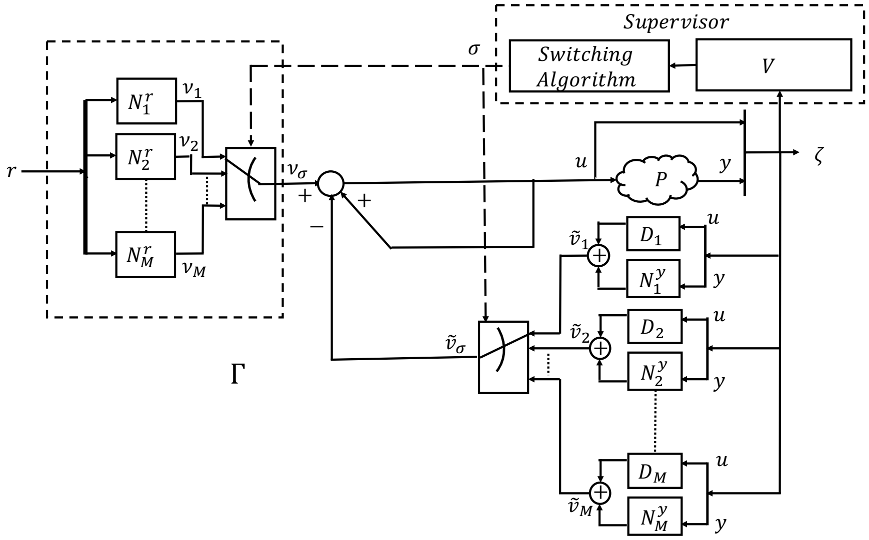

Fig. 4: Switching adaptive control system with MFD realization of controllers

achieved cost expressed as a percentage of the robust cost within a hysteresis constant.

$$\%OS = \frac{\max_t V_\sigma(t) - \Pi^*}{\Pi^*} \times 100\% \tag{15}$$

where $\Pi^* = \Pi_{\text{rsp}} + h$. □

IV. TRANSIENT PERFORMANCE BOUND

In this section, we calculate a theoretical upper bound on the achieved cost given by (14) when the controller cost function given by (13) and hysteresis switching algorithm are used. Subsequently, we calculate a bound on the percent overshoot given by (15). We begin by proving three lemmas. These lemmas are based on the work in [15], [16], wherein several very similar results were derived in conjunction with the different, but theoretically closely related problem of adaptive control resetting.

Lemma 1: Consider the controller realization as shown in Fig. 4. Then,

$$\tilde{v}_{\sigma(t)}(t) = v_{\sigma(t)}(t), \quad \forall t \tag{16}$$

Proof: The following holds from the switching adaptive controller realization of Fig. 4,

$$\begin{aligned} u(t) &= v_{\sigma(t)}(t) - (D_{\sigma(t)} - 1)u(t) - N^y_{\sigma(t)} y(t), && \forall t \\ \Leftrightarrow u(t) &= D_{\sigma(t)}{}^{-1}(v_{\sigma(t)}(t) - N^y_{\sigma(t)} y(t)), && \forall t \end{aligned} \tag{17}$$

The following holds from (10) and (11),

$$\begin{aligned} \tilde{v}_{\sigma(t)}(t) &= \Sigma_{\zeta,\sigma(t)}\zeta(t) \\ &= D_{\sigma(t)}u(t) + N^y_{\sigma(t)} y(t), \quad \forall t \end{aligned} \tag{18}$$

Substituting for u from (17), we get,

$$\tilde{v}_{\sigma(t)}(t) = v_{\sigma(t)}(t), \quad \forall t \tag{19}$$

■

Lemma 2: The following holds for $t \in T_k$,

$$\|\tilde{v}_{\sigma(t_k)}\|_{\lambda p,[0,\,t]} \le \lambda^{t-t_k+1}\|\Sigma_{\zeta,\sigma(t_k)}\zeta\|_{\lambda p,[0,\,t_k-1]} + \|v_{\sigma(t_k)}\|_{\lambda p,[t_k,\,t]} \tag{20}$$

Proof: Since σ remains constant over each switching interval $T_k = \{t_k, t_k+1, \ldots, t_{k+1}-1\}$, the following holds,

$$\sigma(t) = \sigma(t_k), \quad \forall t \in T_k \tag{21}$$

Combining (10), (16) from Lemma 1 and (21), we get,

$$\tilde{v}_{\sigma(t_k)}(t) = \begin{cases} \Sigma_{\zeta,\sigma(t_k)}\zeta(t) & \text{for} \quad t < t_k \\ v_{\sigma(t_k)}(t) & \text{for} \quad t \in T_k \end{cases} \tag{22}$$

From (3), we have for $t \in T_k$,

$$\|\tilde{v}_{\sigma(t_k)}\|_{\lambda p,[0,\,t]} \le \lambda^{t-t_k+1}\|\tilde{v}_{\sigma(t_k)}\|_{\lambda p,[0,\,t_k-1]} + \|\tilde{v}_{\sigma(t_k)}\|_{\lambda p,[t_k,\,t]} \tag{23}$$

Substituting (22) in the above equation we finally get for $t \in T_k$,

$$\|\tilde{v}_{\sigma(t_k)}\|_{\lambda p,[0,\,t]} \le \lambda^{t-t_k+1}\|\Sigma_{\zeta,\sigma(t_k)}\zeta\|_{\lambda p,[0,\,t_k-1]} + \|v_{\sigma(t_k)}\|_{\lambda p,[t_k,\,t]} \tag{24}$$

■

Let,

$$\Pi_k := \min_{K_i \in \mathbb{K}} V_i(t_{k+1} - 1) + h \tag{25}$$

where V_i is given by (13).

Lemma 3: Consider the cost function given by (13). If hysteresis switching algorithm is used, then the following holds for $t \in T_k$,

$$\begin{aligned} \|\zeta\|_{\lambda p,[0,\,t]} \le{}& \lambda^{t-t_k+1}\Pi_k\|\Sigma_{\zeta,\sigma(t_k)}\|_{\lambda p}\|\zeta\|_{\lambda p,[0,\,t_k-1]} \\ &+ \Pi_k[\|v_{\sigma(t_k)}\|_{\lambda p,[t_k,\,t]} + \alpha]. \end{aligned} \tag{26}$$

Proof: From hysteresis switching algorithm, we have,

$$
\begin{aligned}
V_{\sigma(t_k)}(t) &\le \Pi_k, \quad \forall t \in T_k \\
\Leftrightarrow \max_{\tau\in[0,t]} \frac{\|\zeta\|_{\lambda p,[0,\,\tau]}}{\alpha + \|\tilde{v}_{\sigma(t_k)}\|_{\lambda p,[0,\,\tau]}} &\le \Pi_k, \quad \forall t \in T_k \\
\Leftrightarrow \frac{\|\zeta\|_{\lambda p,[0,\,t]}}{\alpha + \|\tilde{v}_{\sigma(t_k)}\|_{\lambda p,[0,\,t]}} &\le \Pi_k, \quad \forall t \in T_k \\
\Leftrightarrow \|\zeta\|_{\lambda p,[0,\,t]} \le \Pi_k[\|\tilde{v}_{\sigma(t_k)}\|_{\lambda p,[0,\,t]} + \alpha&], \quad \forall t \in T_k
\end{aligned} \tag{27}
$$

Using Lemma 2 in the above equation, we get $\forall t \in T_k$,

$$
\begin{aligned}
\|\zeta\|_{\lambda p,[0,\,t]} \le\ & \lambda^{t-t_k+1}\Pi_k\|\Sigma_{\zeta,\sigma(t_k)}\zeta\|_{\lambda p,[0,\,t_k-1]} \\
& + \Pi_k[\|v_{\sigma(t_k)}\|_{\lambda p,[t_k,\,t]} + \alpha]
\end{aligned} \tag{28}
$$

The following holds from (7),

$$
\|\Sigma_{\zeta,\sigma(t_k)}\zeta\|_{\lambda p,[0,\,t_k-1]} \le \|\Sigma_{\zeta,\sigma(t_k)}\|_{\lambda p}\|\zeta\|_{\lambda p,[0,\,t_k-1]} \tag{29}
$$

Therefore, (28) can be written as,

$$
\begin{aligned}
\|\zeta\|_{\lambda p,[0,\,t]} \le\ & \lambda^{t-t_k+1}\Pi_k\|\Sigma_{\zeta,\sigma(t_k)}\|_{\lambda p}\|\zeta\|_{\lambda p,[0,\,t_k-1]} \\
& + \Pi_k[\|v_{\sigma(t_k)}\|_{\lambda p,[t_k,\,t]} + \alpha], \quad \forall t \in T_k
\end{aligned} \tag{30}
$$

■

Rewriting the result of Lemma 3 for $t \in T_{k-1} = \{t_{k-1}, t_{k-1}+1, \ldots, t_k - 1\}$, we get,

$$
\begin{aligned}
\|\zeta\|_{\lambda p,[0,\,t]} \le\ & \lambda^{t-t_{k-1}+1}\Pi_{k-1}\|\Sigma_{\zeta,\sigma(t_{k-1})}\|_{\lambda p}\|\zeta\|_{\lambda p,[0,\,t_{k-1}-1]} \\
& + \Pi_{k-1}[\|v_{\sigma(t_{k-1})}\|_{\lambda p,[t_{k-1},\,t]} + \alpha], \quad \forall t \in T_{k-1}
\end{aligned} \tag{31}
$$

Since $t_k - 1 \in T_{k-1}$, (31) can be written as follows,

$$
\begin{aligned}
\|\zeta\|_{\lambda p,[0,\,t_k-1]} \le\ & \lambda^{t_k-t_{k-1}}\Pi_{k-1}\|\Sigma_{\zeta,\sigma(t_{k-1})}\|_{\lambda p}\|\zeta\|_{\lambda p,[0,\,t_{k-1}-1]} \\
& + \Pi_{k-1}[\|v_{\sigma(t_{k-1})}\|_{\lambda p,[t_{k-1},\,t_k-1]} + \alpha], \quad \forall t \in T_{k-1}
\end{aligned} \tag{32}
$$

Let,

$$x_k = \|\zeta\|_{\lambda p,[0,\,t_k-1]} \tag{33}$$

$$A_{k-1} = \lambda^{t_k-t_{k-1}}\Pi_{k-1}\|\Sigma_{\zeta,\sigma(t_{k-1})}\|_{\lambda p} \tag{34}$$

$$B_{k-1} = \Pi_{k-1} \tag{35}$$

$$u_{k-1} = \|v_{\sigma(t_{k-1})}\|_{\lambda p,[t_{k-1},\,t_k-1]} + \alpha \tag{36}$$

Therefore, (32) can be written as follows:

$$x_k \le A_{k-1}x_{k-1} + B_{k-1}u_{k-1} \tag{37}$$

Let,

$$
\phi(k,l) = \begin{cases} A_{k-1}A_{k-2}\ldots A_l & \text{if} \quad k > l \ge 0 \\ 1 \quad \text{if} \quad k = l \end{cases} \tag{38}
$$

Then, (37) can be written as follows,

$$x_k \le \phi(k,0)x_0 + \sum_{i=0}^{k-1}\phi(k,i+1)B_i u_i \tag{39}$$

The main result of this paper which calculates a bound on the achieved cost of adaptive switching control system and a bound on the percent overshoot is given below.

Theorem 1 (Main Result): Consider the switching adaptive control system shown in Fig. 2 with hysteresis switching algorithm and the λ-cost-detectable cost function given by (13). Let,

$$\Pi^* = \Pi_{\text{rsp}} + h \tag{40}$$

$$\Sigma^* = \max_i \|\Sigma_{\zeta,i}\|_{\lambda p} \tag{41}$$

$$\mu = \Pi^*\Sigma^* \tag{42}$$

where Π_{rsp} is the robust cost as given by Definition 6. Then, the following holds for all $t \in \mathbb{Z}_+$,

$$
V_\sigma(t) \le \begin{cases} \Pi^*\left[\frac{1-\mu^{n_t+1}}{1-\mu}\right] & \text{if} \quad \mu \ne 1 \\ \Pi^*(n_t+1) & \text{if} \quad \mu = 1 \end{cases} \tag{43}
$$

and,

$$
\%OS \le \begin{cases} \mu\left[\frac{1-\mu^{n_t}}{1-\mu}\right] \times 100\% & \text{if} \quad \mu \ne 1 \\ n_t \times 100\% & \text{if} \quad \mu = 1 \end{cases} \tag{44}
$$

where n_t is the number of controller switches till time t.

Proof: When (32) is written in the form of (39), we get,

$$
\begin{aligned}
\|\zeta\|_{\lambda p,[0,\,t_k-1]} \le & \sum_{i=0}^{k-2}\Big(\prod_{j=i+1}^{k-1}\lambda^{t_{j+1}-t_j}\Pi_j\|\Sigma_{\zeta,\sigma(t_j)}\|_{\lambda p}\Big)\Pi_i[\|v_\sigma\|_{\lambda p,[t_i,\,t_{i+1}-1]} \\
& + \alpha] + \Pi_{k-1}[\|v_\sigma\|_{\lambda p,[t_{k-1},\,t_k-1]} + \alpha]
\end{aligned} \tag{45}
$$

where v_σ is the signal seen at the input to the summing junction in Fig. 4. Substituting (45) in the result of Lemma 3 (26), we get for all $t \in T_k$,

$$
\begin{aligned}
\|\zeta\|_{\lambda p,[0,\,t]} \le\ & \lambda^{t-t_k+1}\Pi_k\|\Sigma_{\zeta,\sigma(t_k)}\|_{\lambda p} \\
& \Big[\sum_{i=0}^{k-2}\Big(\prod_{j=i+1}^{k-1}\lambda^{t_{j+1}-t_j}\Pi_j\|\Sigma_{\zeta,\sigma(t_j)}\|_{\lambda p}\Big)\Pi_i[\|v_\sigma\|_{\lambda p,[t_i,\,t_{i+1}-1]} + \alpha] \\
& + \Pi_{k-1}[\|v_\sigma\|_{\lambda p,[t_{k-1},\,t_k-1]} + \alpha]\Big] + \Pi_k[\|v_\sigma\|_{\lambda p,[t_k,\,t]} + \alpha].
\end{aligned} \tag{46}
$$

The above equation can further be written as,

$$
\begin{aligned}
\|\zeta\|_{\lambda p,[0,\,t]} \le & \sum_{i=0}^{k-1}\Big(\prod_{j=i+1}^{k}\Pi_j\|\Sigma_{\zeta,\sigma(t_j)}\|_{\lambda p}\Big)\Pi_i\lambda^{t-t_{i+1}+1}[\|v_\sigma\|_{\lambda p,[t_i,\,t_{i+1}-1]} + \alpha] \\
& + \Pi_k[\|v_\sigma\|_{\lambda p[t_k,\,t]} + \alpha], \quad \forall t \in T_k
\end{aligned} \tag{47}
$$

It can easily be shown that the following holds for all $t \in T_k$ and $i \in [0, k-1]$,

$$\|v_\sigma\|_{\lambda p,[t_i,\,t_{i+1}-1]} \le \lambda^{t_{i+1}-t-1}\|v_\sigma\|_{\lambda p,[0,\,t]} \tag{48}$$

$$\|v_\sigma\|_{\lambda p,[t_k,\,t]} \le \|v_\sigma\|_{\lambda p,[0,\,t]} \tag{49}$$

Therefore, (47) can be written as follows,

$$
\begin{aligned}
\|\zeta\|_{\lambda p,[0,\,t]} \le & \sum_{i=0}^{k-1}\Big(\prod_{j=i+1}^{k}\Pi_j\|\Sigma_{\zeta,\sigma(t_j)}\|_{\lambda p}\Big)\Pi_i[\|v_\sigma\|_{\lambda p,[0,\,t]} + \alpha] \\
& + \Pi_k[\|v_\sigma\|_{\lambda p,[0,\,t]} + \alpha]
\end{aligned} \tag{50}
$$

where $t \in T_k$.

Given (41), (42) and that $\Pi_j \le \Pi^*$ for all j, the following holds,

$$\begin{aligned}\|\zeta\|_{\lambda p,[0,t]} &\le \Pi^*[\|v_\sigma\|_{\lambda p,[0,t]} + \alpha] \\ &+ \Pi^*[\|v_\sigma\|_{\lambda p,[0,t]} + \alpha] \sum_{i=0}^{k-1} (\prod_{j=i+1}^{k} \Pi^*\Sigma^*) \\ &= \Pi^*[\|v_\sigma\|_{\lambda p,[0,t]} + \alpha] \\ &+ \Pi^*[\|v_\sigma\|_{\lambda p,[0,t]} + \alpha] \sum_{i=0}^{k-1} (\prod_{j=i+1}^{k} \mu) \quad \forall t \in T_k \\ &= \Pi^*[\|v_\sigma\|_{\lambda p,[0,t]} + \alpha] \sum_{i=0}^{k} \mu^i\end{aligned} \tag{51}$$

The following is true about the summation term in the above equation,

$$\sum_{i=0}^{k} \mu^i = \begin{cases} \frac{1-\mu^{k+1}}{1-\mu} & \text{if} \quad \mu \neq 1 \\ k+1 & \text{if} \quad \mu = 1 \end{cases} \tag{52}$$

Therefore, Equation (51) can be written as follows,

$$\frac{\|\zeta\|_{\lambda p,[0,t]}}{\alpha + \|v_\sigma\|_{\lambda p,[0,t]}} \le \begin{cases} \Pi^*\left[\frac{1-\mu^{k+1}}{1-\mu}\right] & \text{if} \quad \mu \neq 1 \\ \Pi^*(k+1) & \text{if} \quad \mu = 1 \end{cases} \tag{53}$$

where k represents the number of switches until time $t \in T_k$. Therefore for any $t \in Z_+$, the following holds true,

$$\frac{\|\zeta\|_{\lambda p,[0,t]}}{\alpha + \|v_\sigma\|_{\lambda p,[0,t]}} \le \begin{cases} \Pi^*\left[\frac{1-\mu^{n_t+1}}{1-\mu}\right] & \text{if} \quad \mu \neq 1 \\ \Pi^*(n_t+1) & \text{if} \quad \mu = 1 \end{cases} \tag{54}$$

$$\Leftrightarrow V_\sigma(t) \le \begin{cases} \Pi^*\left[\frac{1-\mu^{n_t+1}}{1-\mu}\right] & \text{if} \quad \mu \neq 1 \\ \Pi^*(n_t+1) & \text{if} \quad \mu = 1 \end{cases} \tag{55}$$

where n_t is the number of controller switches till time t. Since (55) holds for all $t \in \mathbb{Z}_+$, the following holds,

$$\max_t V_\sigma(t) \le \begin{cases} \Pi^*\left[\frac{1-\mu^{n_t+1}}{1-\mu}\right] & \text{if} \quad \mu \neq 1 \\ \Pi^*(n_t+1) & \text{if} \quad \mu = 1 \end{cases} \tag{56}$$

Substituting the above in (15), we get,

$$\%OS \le \begin{cases} \mu\left[\frac{1-\mu^{n_t}}{1-\mu}\right] \times 100\% & \text{if} \quad \mu \neq 1 \\ n_t \times 100\% & \text{if} \quad \mu = 1 \end{cases} \tag{57}$$

■

Of the parameters that appear in the bound above, Σ^* in μ and n_t which is bounded above by n_s [1] can be changed by the designer. In the next section, we discuss the implications of changing each of these designer controlled parameters on the transient performance of the system.

V. SIMULATION RESULTS

A. Bad Transients

Suppose the unknown plant shown in Fig. 2 has a transfer function, $P(z) = \frac{(e^{0.03}-1)z}{z-e^{0.03}}$, which is a discretized version of the plant used in [14], with a sampling interval of 0.03 seconds and the controller set is $\mathbb{K} = \{K_1 = -2, K_2 = 2\}$, where $N_1^r = N_1^y = -2$, $N_2^r = N_2^y = 2$ and $D_1 = D_2 = 1$. If the reference signal is chosen to be $r(t) = 1, \forall t$, the hysteresis algorithm *A1* is used with $h = 0.01$, $\hat{K}(0) = K_1$ and the cost function (13) is used with $\lambda = 0.99$, $p = 2$ and $\alpha = 1$, the simulation results show that $n_t = 9$. It can be verified[2] that $\Sigma^* = 2$, $\Pi_{\text{rsp}} \approx 1.81$, therefore, $\Pi^* = \Pi_{\text{rsp}} + h \approx 1.82$ and $\mu \approx 3.6$. Therefore, the theoretical bound on $\%OS$ given by (57) is ,

$$\mu\left[\frac{1-\mu^{n_t}}{1-\mu}\right] \times 100\% \approx 1.4 \times 10^7\%.$$

The simulation results are shown in Fig. 5 which indicate large transients, $\max_t |u(t)| \approx 175$ and $\max_t |y(t)|$.

B. Reducing n_s

Based on Theorem 1, one strategy to reduce the overshoot is to modify the hysteresis constant h, so as to reduce the upper bound on the number of controller switches n_s given by (12). The bound on n_s can be drastically reduced by increasing the hysteresis constant h. If $h = \Pi_{\text{rsp}}$, then, for the example given in the previous subsection, we have

$$n_s = \left\lceil M \frac{\Pi_{\text{rsp}}}{h} \right\rceil = \left\lceil 2 \frac{\Pi_{\text{rsp}}}{\Pi_{\text{rsp}}} \right\rceil = 2$$

If all other parameters are the same as in the previous subsection, we get $\Pi^* = \Pi_{\text{rsp}} + h \approx 3.62$ and $\mu = 7.24$. Simulation results show that $n_t = 1$. Therefore, the theoretical bound on $\%OS$ given by (57) is ,

$$\mu\left[\frac{1-\mu^{n_t}}{1-\mu}\right] \times 100\% \approx 7.24 \times 100\%.$$

The simulation results are shown in Fig. 6 and it can be seen that the transients have reduced significantly.

C. Reducing Σ^*

The bound in (57) can be reduced further by choosing different co-prime realizations N_i, D_i for the candidate controllers K_i, which in turn reduces the transients. If each of the controllers in $\mathbb{K}$ of (V-A) is realized using the following co-prime factor descriptions,

$$\begin{aligned} N_1^r(z) = N_1^y(z) &= \frac{-2(0.33z - 0.26)}{z - 0.86} \\ N_2^r(z) = N_2^y(z) &= \frac{2(0.33z - 0.26)}{z - 0.86} \\ D_1(z) = D_2(z) &= \frac{0.33z - 0.26}{z - 0.86} \end{aligned}$$

then, $\Sigma^* = 1.04$, $\Pi_{\text{rsp}} \approx 3.5$, therefore $\Pi^* = \Pi_{\text{rsp}} + h \approx 5.31$ and $\mu = 5.52$. The bound on $\%OS$ in (57) is,

$$\mu\left[\frac{1-\mu^{n_t}}{1-\mu}\right] \times 100\% \approx 5.52 \times 100\%.$$

The simulation results with the reduced Σ^* are shown in Fig. 7 and it can be seen that the transients have reduced further when compared to that in (V-B).

[1] n_s is the upper bound on the total number of controller switches while n_t is the number of switches till time t

[2] The $\ell_{\lambda p}$ gain of an LTI system Γ, *i.e.*, $\|\Gamma(z)\|_{\lambda p}$ is equivalent to the ℓ_p gain of $\Gamma(\lambda z)$. When $p = 2$, the ℓ_2 gain and H_∞ gain are equal as shown in [20], [21]. Therefore, the MATLAB command *norm* can be used to calculate the $\ell_{\lambda 2}$ gain as $\|\Gamma(z)\|_{\lambda 2} = \text{norm}(\Gamma(\lambda z), \text{Inf})$.

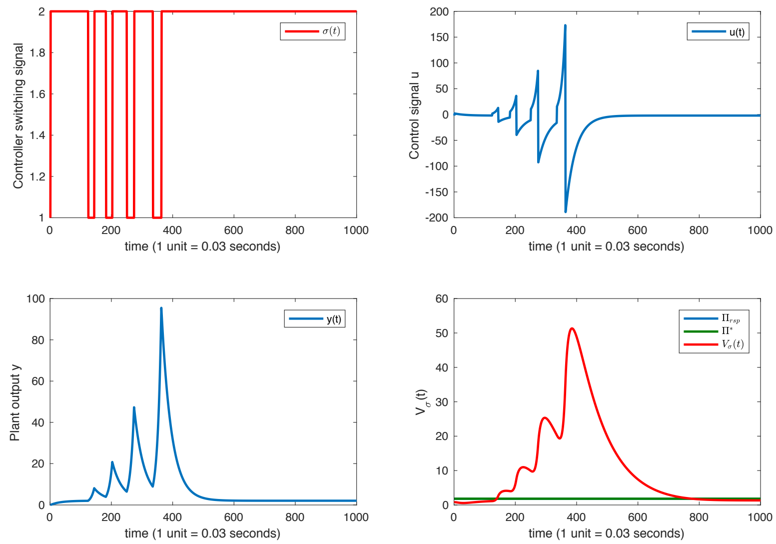

Fig. 5: Case A (Bad transients). *Top-left*: Controller switching signal σ, *Top-right*: Control signal u, *Bottom-left*: Plant o/p signal y, *Bottom-right*: Achieved cost V_σ , Robust cost Π_{rsp} and Π^*

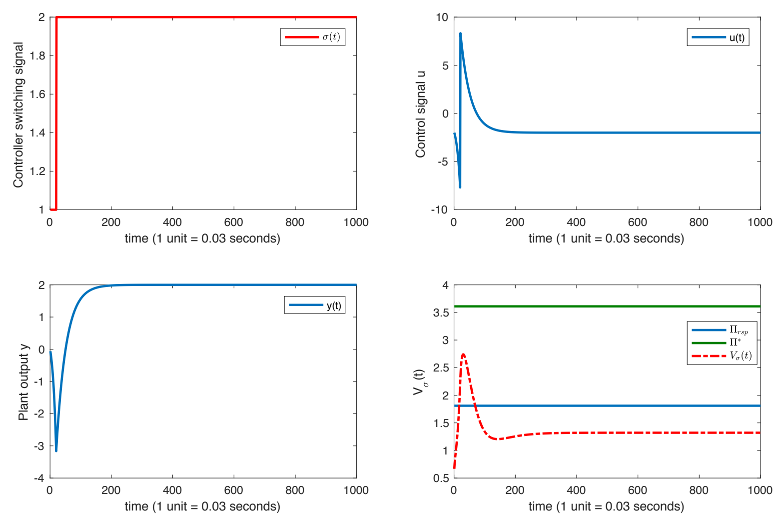

Fig. 6: Case B (With improvements). *Top-left*: Controller switching signal σ, *Top-right*: Control signal u, *Bottom-left*: Plant o/p signal y, *Bottom-right*: Achieved cost V_σ , Robust cost Π_{rsp} and Π^*

VI. CONCLUSIONS

A transient performance analysis for the class of adaptive control systems with hysteresis switching algorithm and ℓ_2 gain type cost detectable cost functions was proposed. A theoretical upper bound on the percent overshoot of the achieved cost with respect to robust cost was obtained and the parameters that can reduce this bound were highlighted. Simulation results were provided that indicated improvement in the transient performance with the proposed ideas.

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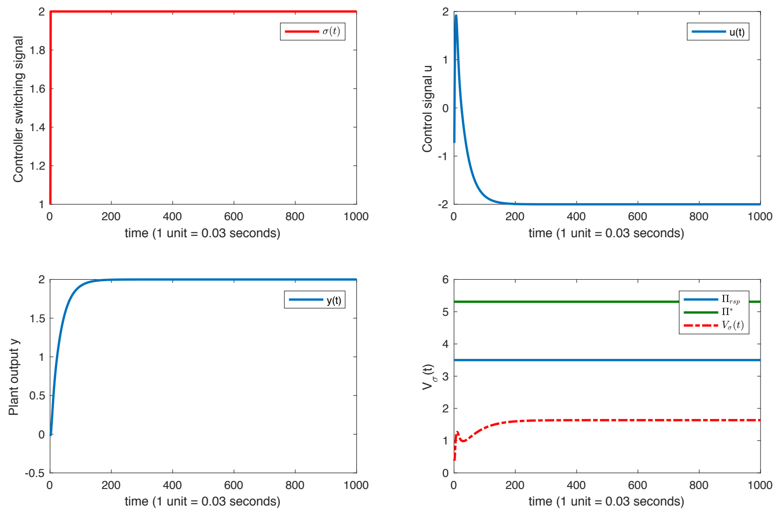

Fig. 7: Case C (With improvements). *Top-left*: Controller switching signal σ, *Top-right*: Control signal u, *Bottom-left*: Plant o/p signal y, *Bottom-right*: Achieved cost V_σ , Robust cost $\Pi_{\rm rsp}$ and Π^*

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