Tracking Trojan Asteroids in Periodic and Quasi-Periodic Orbits around the Jupiter Lagrange Points using LDV Techniques

Fariba Ariaei and Edmond Jonckheere
Dept. of Electrical Eng.
University of Southern California
Los Angeles, CA 90089
ariaei@usc.edu, jonckhee@usc.edu

Stephan Bohacek
Dept. of Electrical and Comp. Eng.
University of Delaware
Newark, DE 19716
bohacek@eecis.udel.edu

Abstract

Linear Dynamically Varying (LDV) control is a technique for getting a natural (nonlinear, possibly chaotic) trajectory and a perturbed trajectory asymptotically synchronized given an initial condition offset. Probably the best illustrative example, which motivates this paper, is tracking the natural (possibly quasi-periodic) motion of a Trojan asteroid near the L4 point of Jupiter with a spacecraft that follows a trajectory perturbed by the non conservative propulsion forces. The tracking error is linearized around the natural dynamics of the Trojan body, leading to an LDV model of the tracking error. This in turn leads to a dynamically varying controller, itself given as the solution to a Partial Differential Riccati Equation, solved via the method of characteristics. It is shown that this technique allows for accurate tracking of the complicated dynamics around the L4 point.

1 Introduction

In this paper, we describe an astrodynamical application of a general synchronization technique that was developed for "complicated," possibly chaotic, dynamics and that can be described as follows: Let

$$\dot{\theta}(t) = f(\theta(t))$$

be a natural dynamics while

$$\dot{\phi}(t) = f(\phi(t), u(t))$$

is a perturbed dynamics, by which we mean that $f(\phi, 0) = f(\phi)$, the two dynamics being subjected to an initial condition offset, viz., $\theta(0) \neq \phi(0)$. The problem is to design a controller $u(t)$ such that $\phi(t) - \theta(t) \to 0$ as $t \to \infty$. Probably the best illustrative example of this problem formulation is when $\theta(t)$ is thought to be the state of such a light body as an asteroid, a Trojan body, or a comet, while $\phi(t)$ is the state of a spacecraft in a rendezvous mission with the body. In the case of a Trojan, the attractions of Jupiter and the sun are dominant compared with the mutual Trojan-spacecraft attraction, so that both the Trojan and the spacecraft would be subject to the same dynamics–the restricted 3 body problem dynamics–provided the spacecraft does not use its nonconservative propulsion force $u(t)$. While this paper focuses on the Trojan body around the libration Lagrange L4 point, the same can be said about the Wirtanen comet. In fact, with a diameter of just a few miles (compared to a few hundred miles for a Trojan), the mutual attraction is so weak that the Rosetta lander will have to be anchored to the Wirtanen comet.

The proposed method of solution is of the Linear Dynamically Varying (LDV) type, that is, the tracking error dynamics is linearized around the nominal dynamics as $\dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t)$. It is important to observe that the coefficient matrices depend on the state of the nominal dynamics and hence are *dynamically varying*. The controller is designed on the basis of the linearized error and is also of the dynamically varying type [1].

2 Conservative and Nonconservative Lagrangian Dynamics

We briefly review the Hamiltonian dynamics of the natural motion of a Trojan asteroid, later to be amended so as to incorporate the nonconservative dynamics of a spacecraft subject to the same gravitational field as the Trojan.

2.1 Conservative Dynamics

The Hamiltonian function of a massless body in the Sun-Jupiter gravitational attraction in the absence of nonconservative forces is given by [1]:

$$
\begin{aligned}
H &= \frac{1}{2}(P_\xi^2 + P_\eta^2) + \omega(\eta P_\xi - \xi P_\eta) \\
&\quad -\omega \dot{X}_{L_4}(Y_{L_4} + \eta) + \omega \dot{Y}_{L_4}(X_{L_4} - \xi) \\
&\quad -\frac{1}{2}\dot{X}_{L_4}^2 - \frac{1}{2}\dot{Y}_{L_4}^2 - \frac{1}{2}\omega^2 Y_{L_4}^2 \\
&\quad -\frac{1}{2}\omega^2 \dot{X}_{L_4}^2 + \omega^2(X_{L_4}\xi - Y_{L_4}\eta) \\
&\quad -\dot{\omega} Y_{L_4}\xi - \omega \dot{Y}_{L_4}\xi - \dot{\omega} X_{L_4}\eta - \ddot{X}_{L_4}\xi \\
&\quad -\omega \dot{X}_{L_4}\eta + \ddot{Y}_{L_4}\eta - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}
\end{aligned} \tag{1}
$$

where

$$
\begin{aligned}
r_1 &= \sqrt{(\xi - X_{L_4} - \mu)^2 + (\eta + Y_{L_4})^2} \\
r_2 &= \sqrt{(\xi - X_{L_4} + 1 - \mu)^2 + (\eta + Y_{L_4})^2}
\end{aligned}
$$

These equations are written in the *non-inertial* frame centered at the L_4 point, with its $\vec{i}$-axis aligned with the Sun-Jupiter axis and its $\vec{j}$-axis orthogonal to the $\vec{i}$-axis. (ξ, η) is the coordinate vector of the third body in the $\left(L_4, \vec{i}, \vec{j}\right)$ frame, and (P_ξ, P_η) is the momentum vector associated with (ξ, η). (X_{L_4}, Y_{L_4}) are the coordinates of the L_4 point in the frame $\left(C.G., \vec{i}, \vec{j}\right)$ centered at the center of gravity (C.G.) of the Jupiter-Sun system and with its axes aligned with the $(\vec{i}, \vec{j})$-axes. The parameter ω denotes the angular velocity of the $\left(C.G., \vec{i}, \vec{j}\right)$ frame associated with the primaries. The physical units are normalized so that the semi-major axis of the Sun-Jupiter orbit is set to 1, the total mass of the Sun-Jupiter system is set to 1, with μ the mass of the smaller primary body, and the universal gravitation constant is also set to 1. The latter implies that, if the eccentricity $e = 0$, ω is set to 1, equivalently, the period of Jupiter is set to 2π. With these conventions, $r_1$and r_2 are the distance functions between the third body and the massive bodies, m_1 and m_2, respectively [2].

The *unperturbed* Hamiltonian, H_0, that is, that part of the Hamiltonian quadratic in ξ, η, P_ξ, P_η and evaluated for $e = 0$, is examined first and transformed as

$$
H_0 = \sum_{k=1}^{2} \omega_k J_k \tag{2}
$$

where the J_k's are the action variables, that is, the conjugate momenta associated with the coordinates angle variables ϕ_k's. This unperturbed Hamiltonian, which is a function of the action variables only, is in its integrable form.

To investigate the Hamiltonian motion associated with the perturbed Eq.(1), first, the $\frac{1}{r_1}$ and $\frac{1}{r_2}$ terms of the Hamiltonian are expanded up to the sixth power of $\frac{\xi}{X_{L4}}, \frac{\eta}{Y_{L4}}$ and ω is also approximated up to the third power of the eccentricity. Then several canonical transformations are performed [1]. It follows that, in the natural motion of the Trojan body, there is a 1:1 resonance between the natural frequency of the motion of the third body and the mean motion of Jupiter [1]. The final form of the Hamiltonian function under the 1:1 resonance condition is

$$
H(I, \vartheta_1) = H_0(I) + \sum_k H_k(I, \vartheta_1) \tag{3}
$$

where H_k is that part of the Hamiltonian of the kth order in the eccentricity and the I_i's and the ϑ_i's are the new action and the angle variables, respectively. These variables include, in addition to those of the unperturbed motion, the mean anomaly ℓ and the conjugate mean anomaly of the primaries. In the spirit of small perturbation theory, the overall, perturbed, Hamiltonian is nearly integrable. The corresponding equations of motion are

$$
\begin{aligned}
\dot{\vartheta}_1 &= \frac{\partial H}{\partial I_1}, & \dot{I}_1 &= -\frac{\partial H}{\partial \vartheta_1}, \\
\dot{\vartheta}_2 &= \frac{\partial H}{\partial I_2} = 1, & \dot{I}_2 &= -\frac{\partial H}{\partial \vartheta_2} = 0, \\
\dot{\vartheta}_3 &= \frac{\partial H}{\partial I_3}, & \dot{I}_3 &= -\frac{\partial H}{\partial \vartheta_3} = 0.
\end{aligned} \tag{4}
$$

From Eq.(4), I_2 and I_3 are constant along the motion. Therefore, the equations of motion of the third body in the Sun-Jupiter gravitational field involve the variables $\dot{I}_1$ and $\dot{\vartheta}_1$only.

2.1.1 Periodic and Quasi-periodic Orbits:

The parameters of the Sun-Jupiter system in physically meaningful units are the following:

$$
\mu = 0.000954726, \quad e = 0.048, \quad \dot{\ell} = \omega_0 = 1. \tag{5}
$$

Phase plane plots for the averaged perturbed system show existence of a point enclosed by several closed curves, which are obtained for specific initial conditions. This point corresponds to a periodic orbit, and the closed curves correspond to quasi-periodic motions around L_4. Solving Equations4 reveals periodic and quasi-periodic orbits around L_4. The periodic orbits have a period of $T = 1847.72$, which is equal to 3666.75 Earth's years or 308.13 Jupiter's years. The quasi-periodic orbits appear Lyapunov stable for that period of time [3]. Furthermore, computer simulations reveal that there exists a compact invariant set $\Theta \supseteq L_4$.

2.2 Nonconservative Hamiltonian Dynamics

In the problem of steering a spacecraft to a pursuit of a Trojan body in one of its known orbits, the spacecraft is considered as another third body in the Sun-Jupiter system, but this one is under the influence of a nonconservative propulsion force. This nonconservative force is used to direct the spacecraft towards the desired trajectory.

The Lagrange equations of motion, in the case of nonconservative forces, can be written as follows:

$$\frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{q\dot{q}}\right] - \left(\frac{\partial L}{\partial q_i}\right)_{q\dot{q}} = F \cdot \left(\frac{\partial r}{\partial q_i}\right)_q \quad (6)$$

where F is the nonconservative force, $L = T - U$ is the Lagrange function, that is, the sum of the kinetic energy and the potential energy of the conservative force, $r = r(q) = r(q_1, q_2, ..., q_n)$ is the body position vector, and the q_is are the generalized coordinates. Hamilton's equations of motion are

$$\begin{aligned} \dot{p}_i &= -\left(\frac{\partial H}{\partial q_i}\right)_{qpt} + F \cdot \left(\frac{\partial r}{\partial q_i}\right)_q, \\ \dot{q}_i &= \left(\frac{\partial H}{\partial p_i}\right)_{qpt}. \end{aligned} \quad (7)$$

In case there is a need for a canonical transformation on this system, it can be shown that the new Hamiltonian in the new canonical variables $\{Q, P\}$ is as follows:

$$K(Q, P, t) = \sum_{i=1}^{n} P_i \dot{Q}_i - L(Q, \dot{Q}, P, \dot{P}, t) + \frac{dG}{dt} \quad (8)$$

where $G = G(q, Q, t)$ is the generating function that transforms the old canonical variables, $\{q, p\}$, to the new ones, $\{Q, P\}$. (Observe that our choice of generating function is independent of F). It can be shown that the transformed Hamiltonian K does not either directly incorporate the force F [4]. Therefore, via the Lagrange modified equations, the Hamilton modified equations can be obtained as

$$\begin{aligned} \dot{P}_i &= -\left(\frac{\partial K}{\partial Q_i}\right)_{QPt} + F \cdot \left(\frac{\partial r}{\partial Q_i}\right)_{QPt}, \\ \dot{Q}_i &= \left(\frac{\partial K}{\partial P_i}\right)_{QPt} - F \cdot \left(\frac{\partial r}{\partial P_i}\right)_{QPt}. \end{aligned} \quad (9)$$

2.2.1 Non-conservative Spacecraft in Restricted Third Body Dynamics: Now, consider Fig.(1), in which the third body, as it is called in the three-body problem, is a spacecraft influenced by nonconservative propulsion forces. These forces are applied in the ξ and η directions to control the spacecraft to track the desired orbit in the $(C.G., \vec{i}, \vec{j})$ coordinate system. The nonconservative force and the position of the third body can be rewritten, respectively, as:

$$F = F_\xi \, \vec{i} + F_\eta \, \vec{j} \quad (10)$$

$$r = (\xi - X_{L_4}) \, \vec{i} + (\eta + Y_{L_4}) \, \vec{j} \quad (11)$$

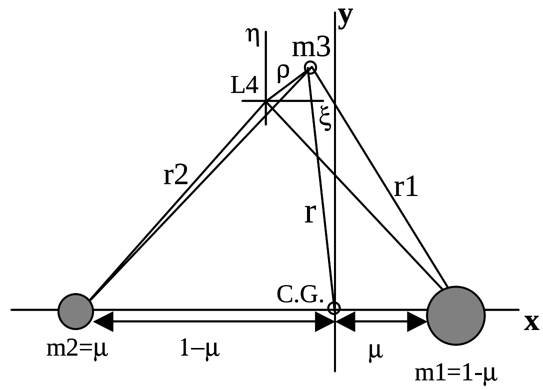

Figure 1: The elliptic restricted three-body problem.

Therefore, using Eq.(7), the nonconservative Hamilton equations of motion of the spacecraft, in the physically motivated coordinates, can be obtained, respectively, as

$$\begin{aligned} \dot{\xi} &= \frac{\partial H}{\partial P_\xi}, \\ \dot{P}_\xi &= -\frac{\partial H}{\partial \xi} + F(\frac{\partial r}{\partial \xi}) = -\frac{\partial H}{\partial \xi} + F_\xi, \\ \dot{\eta} &= \frac{\partial H}{\partial P_\eta}, \\ \dot{P}_\eta &= -\frac{\partial H}{\partial \eta} + F(\frac{\partial r}{\partial \eta}) = -\frac{\partial H}{\partial \eta} + F_\eta. \end{aligned} \quad (12)$$

H is the Hamiltonian function for the perturbed Hamiltonian system of the three bodies

As for the conservative case, the restricted three-body problem cannot be easily solved using the above equations of motion. Therefore, as in the conservative case, several canonical transformations are applied to simplify the Hamiltonian function, H, to its nearly integrable form [1]. Finally, using Eq.(9), the Hamilton equations of motion for the spacecraft, which is influenced by the gravitational forces of the sun and Jupiter and the nonconservative force F, can be obtained as follows:

$$\begin{aligned} &\dot{\vartheta}_1 = \frac{\partial H^*}{\partial I_1} - F(\frac{\partial r}{\partial I_1}), \; \dot{I}_1 = -\frac{\partial H^*}{\partial \vartheta_1} + F(\frac{\partial r}{\partial \vartheta_1}), \\ &\dot{\vartheta}_2 = 1 - F(\frac{\partial r}{\partial I_2}), \qquad \dot{I}_2 = F(\frac{\partial r}{\partial \vartheta_2}), \\ &\dot{\vartheta}_3 = \frac{\partial H^*}{\partial I_3} - F(\frac{\partial r}{\partial I_3}), \; \dot{I}_3 = F(\frac{\partial r}{\partial \vartheta_3}). \end{aligned} \quad (13)$$

where H^* is the transformed Hamiltonian in the action-angle variables (ϑ_i, I_i), where

$$r = r(\xi(I_1, I_2, I_3, \vartheta_1, \vartheta_2, \vartheta_3), \eta(I_1, I_2, I_3, \vartheta_1, \vartheta_2, \vartheta_3)).$$

3 Tracking a Trojan Asteroid Using LDV Control

3.1 LDV Systems

The tracking error is defined as $x(t)$:

$$x(t) = \phi(t) - \theta(t) \tag{14}$$

and the rate of change of the error is:

$$\begin{aligned}\dot{x}(t) &= \dot{\phi}(t) - \dot{\theta}(t) \\ &= f(\phi(t), u(t)) - f(\theta(t), 0)\end{aligned} \tag{15}$$

Taylor series expansion around the point $(\theta(t), 0)$ is now applied to the of equation. Therefore the linear part of the expansion is as follows [5]:

$$\begin{aligned}\dot{x}(t) &= \left(\frac{\partial f(\phi, u)}{\partial \phi}\right)_{\theta(t),0} (\phi(t) - \theta(t)) \\ &\quad + \left(\frac{\partial f(\phi, u)}{\partial u}\right)_{\theta(t),0} u(t) \\ &= A_{\theta(t)}x(t) + B_{\theta(t)}u(t)\end{aligned} \tag{16}$$

Definition 1 *The LDV system $\frac{d}{dt}x(t) = A_{\theta(t)}x(t)$ is said to be exponentially stable (along the flow of f) if there exist functions $\alpha(\theta_0) \in (0, \infty)$ and $\beta(\theta_0) \in [0, \infty)$ such that, for every $x_0 \in \Theta$,*

$$||x(t)|| \leq \beta(\theta_0)e^{-\alpha(\theta_0)t}||x(0)||$$

where $||x|| = \sqrt{x^T x}$. The system is said to be uniformly exponentially stable iff there are numbers $\alpha \in (0, \infty)$, $\beta \in [0, \infty)$ such that, for every $\theta \in \Theta$,

$$||x(t)|| \leq \beta e^{-\alpha t}||x(0)||$$

Finally, the system is said to be asymptotically stable iff, for every $\theta \in \Theta$,

$$\lim_{t\to\infty} ||x(t)|| = 0$$

Remark 1 *Observe that stability and exponential stability are properties of the various trajectories, whereas uniform asymptotic stability is a property of the flow.*

Proposition 1 *If the LDV system $\frac{d}{dt}x(t) = A_{\theta(t)}x(t)$ is continuous and $\theta(t)$ runs over a compact set Θ, then asymptotic stability, exponential stability, and uniformly exponential stability are equivalent.*

Definition 2 *The LDV system $\frac{d}{dt}x = A_{\theta(t)}x + B_{\theta(t)}u$ is said to be stabilizable iff there exists a function $K : \Theta \times \mathbb{R}^+ \to \mathbb{R}^p$ such that $\frac{d}{dt}x(t) = (A_{\theta(t)} + B_{\theta(t)}K_{\theta(t)}(t))x(t)$ is (uniformly) asymptotically stable.*

Definition 3 *The LDV system $\frac{d}{dt}x(t) = A_{\theta(t)}(t)x(t)$, $z_1(t) = C_{\theta(t)}x(t)$ is said to be detectable if there exists an output injection feedback $L : \Theta \times \mathbb{R}^+ \to \mathbb{R}^n$ such that $\frac{d}{dt}x(t) = (A_{\theta(t)} + L_{\theta(t)}(t)C_{\theta(t)})x(t)$ is (uniformly) asymptotically stable.*

Remark 2 *Observe that stabilizability and detectability are weak conditions; continuity in θ of the feedback matrices K, L is not required and furthermore they are allowed to be explicit functions of the time t. It is a nontrivial result that, despite the weakness of the stabilizability condition, it is sufficient to guarantee that LQ control yields a continuous feedback, not explicitly depending on the time. Regarding detectability, it is because a continuous-time dynamics is reversible that such a weak form of detectability is sufficient. In contrast, because discrete-time dynamics need not be reversible, a more stringent detectability condition is needed in that case. See p. 842 of [5]*

3.2 LDV Control

Now, we are in a position to state the main theorem.

Theorem 2 *Assume that the functions A_θ, B_θ, $Q_\theta = C_\theta^T C_\theta$, R_θ are continuous and such that (A, C) is detectable and $R > 0$. Then (A, B) is stabilizable iff there exists a continuous map*

$$X : \Theta \to \mathbb{R}^{n\times n}, X_\theta \geq 0,$$

differentiable along the trajectories of f, that satisfies the Partial Differential Riccati Equation (PDRE),

$$\begin{aligned}&\sum_{i=1}^{n} \frac{\partial X_\theta}{\partial \theta_i} f^i(\theta) + A_\theta^T X_\theta + X_\theta A_\theta \\ &\quad + Q_\theta - X_\theta B_\theta R_\theta^{-1} B_\theta^T X_\theta = 0\end{aligned} \tag{17}$$

and such that $A_\theta - B_\theta R^{-1} B_\theta^T X_\theta$ is uniformly asymptotically stable. Furthermore,

$$\begin{aligned}&x_0' X_{\theta_0} x_0 = \\ &\quad \inf_u \int_0^\infty \left(x(\tau)^T Q_{\theta(\tau)} x(\tau) + u(\tau)^T R_{\theta(\tau)} u(\tau)\right) d\tau\end{aligned}$$

Proof: The proof follows from a continuous-time adaptation of the argument of Th. 1 of [5], except for one issue: Continuity of X_θ defined as the cost matrix follows the same argument as in the discrete-time case, but the fact that it satisfies the PDRE stems from the observation that $\sum_{i=1}^{n} \frac{\partial X_\theta}{\partial \theta_i} f^i(\theta)$ is $\frac{d}{dt}X_{\theta(t)}$ evaluated along trajectories, from which it is obvious that X_θ is differentiable along trajectories. (We conjecture that it is differentiable across trajectories, but the proof has eluded us.) ■

Remark 3 *Some systems, like the Trojans, have sensitive dependence on initial conditions. Despite the sensitive dependence of the trajectory on θ_0, the cost to stabilize the trajectory remains continuous.This is the counterintuitive fact of this theorem; see Remark 2 of [5].*

The PDRE is numerically solved by what amounts to the method of characteristics. Again, from the observation that $\sum_{i=1}^{n} \frac{\partial X_\theta}{\partial \theta_i} f^i(\theta(t)) = \frac{d}{dt} X_{\theta(t)}$, we solve the partial differential equation along the trajectories of f as the differential equation $\frac{d}{dt} X_{\theta(t)} = -A_{\theta(t)}^T X_{\theta(t)} - X_{\theta(t)} A_{\theta(t)} - Q_{\theta(t)} + X_{\theta(t)} B_{\theta(t)} R_{\theta(t)}^{-1} B_{\theta(t)}^T X_{\theta(t)}$. More specifically, assume we want to evaluate X_{θ_0} somewhere along a trajectory to be tracked. Let $\theta_0 = \theta(o)$. Anticipate the motion of the trajectory over $[0, T]$. Set the terminal condition $X(T, T)$, integrate backward the Riccati differential equation

$$\begin{aligned} \frac{d}{dt} X(t,T) &= -(A_{\theta(t)}^T X(t,T) + X(t,T) A_{\theta(t)} + Q_{\theta(t)}) \\ &\quad + X(t,T) B_{\theta(t)} R_{\theta(t)}^{-1} B_{\theta(t)}^T X(t,T) \end{aligned}$$

and set $X_{\theta_0} = X(0, T)$. This numerical scheme somehow provides a test of the stabilizability condition, which is hard to check in practice. Should the backward integration reveal a solution growing without bound, then there is evidence that the stabilizability (or detectability) condition fails.

With the LDV gain at hand, we can state the following:

Theorem 3 *Under the same conditions as the preceding theorem, there exists a neigbohorhood U of the natural trajectory such that, if $\dot{\phi}(t) = f(\phi(t), -R_{\theta(t)}^{-1} B_{\theta(t)}^T X_{\theta(t)} (\phi(t) - \theta(t)))$ and $\phi(0) \in U$,then $\phi(t) - \theta(t) \to 0$ as $t \to \infty$.*

3.3 Three Body Tracking by LDV Method

Tracking a desired orbit around the libration point, L_4, in the Sun-Jupiter-Trojan system, is achieved by the LDV method. For this system, the controller generates a nonconservative force, F, applied to the spacecraft to correct its position and to direct it toward the desired trajectory. The natural trajectory of the Trojan is given by Eq.(4) and the motion of the spacecraft, perturbed under the force F, is given by Eq.(13). Since the equations of motion of the third body in the three body problem is highly nonlinear, a numerical method, Runge-Kutta 4, is applied for finding the solution of the Hamilton equations of motion.

3.3.1 Tracking the Periodic Orbits of the Trojan Asteroid: The unperturbed Hamiltonian in the physically motivated coordinates is obtained as follows [1]:

$$\begin{aligned} H_0 &= \frac{1}{2}(P_\xi^2 + P_\eta^2) + \omega_0(\eta P_\xi - \xi P_\eta) \\ &\quad + A\xi^2 + B\xi\eta + C\eta^2 \end{aligned} \tag{18}$$

where the coefficients A, B and C are set to 0.125, 1.296 and -0.65, respectively. Using Eq.(12), the Hamiltonian equations of motion of the spacecraft are obtained as follows:

$$\dot{\phi} = \begin{cases} \dot{\xi} = P_\xi + \omega_0 \eta, \\ \dot{\eta} = P_\eta - \omega_0 \xi, \\ \dot{P_\xi} = \omega_0 P_\eta - 2A\xi - B\eta + F_\xi, \\ \dot{P_\eta} = -\omega_0 P_\xi - 2C\eta - B\xi + F\eta. \end{cases} \tag{19}$$

$$u(t) = \begin{bmatrix} F_\xi(t) \\ F_\eta(t) \end{bmatrix} \tag{20}$$

The tracking error of the periodic orbit of the Trojan asteroid is obtained as the solution to:

$$\dot{x} = \begin{bmatrix} 0 & \omega_0 & 1 & 0 \\ -\omega_0 & 0 & 0 & 1 \\ -2A & -B & 0 & \omega_0 \\ -B & -2C & -\omega_0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

After solving the partial differential Riccati equation, Eq.(17), and simulating the motion, the tracking trajectories and the control inputs are obtained as those shown in Fig(2), Fig(3) and Fig(4).

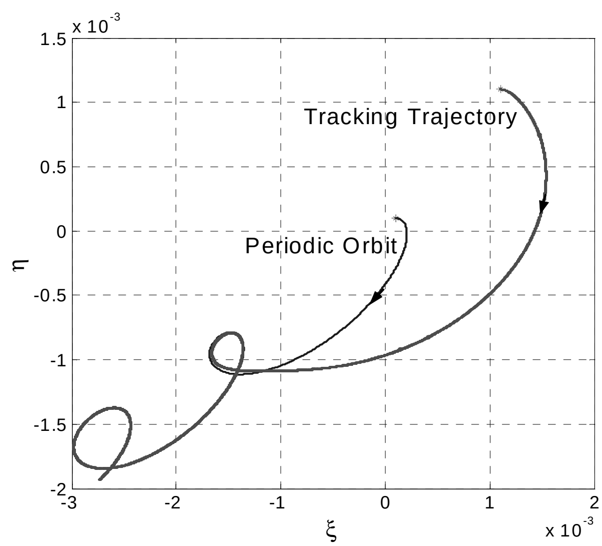

Figure 2: Periodic orbit of the Trojan and the spacecraft's tracking trajectory.

3.3.2 Tracking the Quasi-Periodic Orbits of the Trojan Asteroid: Because the full perturbed

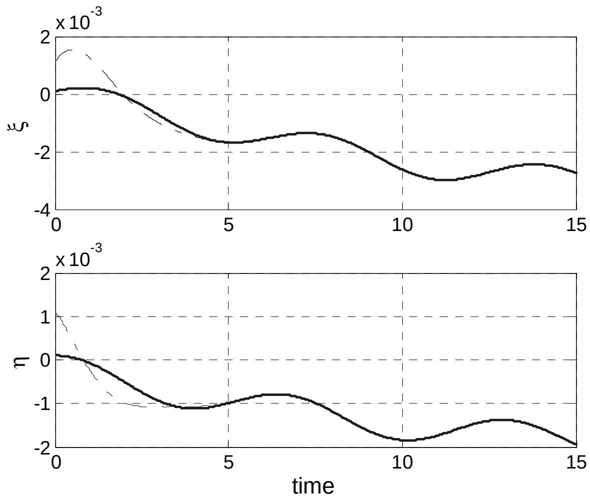

Figure 3: Desired (solid line) and tracking (dashed line) trajectories in xi-eta directions.

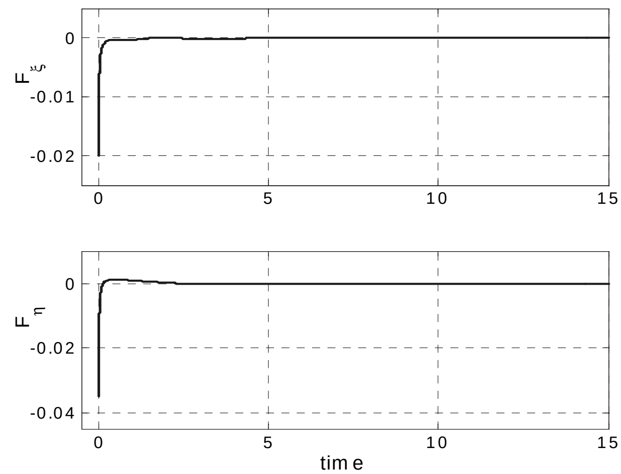

Figure 4: The control inputs of the spacecraft in xi-eta directions.

equations in the canonical variables are complicated, we focused on the decoupled (ϑ_1, I_1) equations. Further, we average the natural motion of the Trojan. However, we cannot average the nonconservative motion of the spacecraft because the fast variables of

$$\frac{\partial r}{\partial \vartheta_1}, \frac{\partial r}{\partial I_1}$$

happen to vanish. Despite the discrepancy between the natural and the forced trajectories, computer simulations seem to indicate that the spacecraft tracks the Trojan. Whether or not the tracking will ultimately be proved to happen, the previous observation is definitely in agreement with the proved robustness of LDV design against dynamics uncertainty [6].

4 Conclusions

It has been shown that tracking the, possibly complicated, dynamics of a massless body in the gravitational field of other, massive bodies with a spacecraft is probably the most natural application of Linear Dynamically Varying (LDV) control [5], the key point of which is to achieve synchronization between a natural dynamics and a perturbed dynamics subject to an offset of initial conditions.

References

[1] Fariba Ariaei, "Periodic and quasi-periodic orbits around the l4 lagrangian point in the elliptic restricted three-body problem," M.S. thesis, Sharif University of Technology, 1999.

[2] William E. Wiesel, *Spaceflight Dynamics*, McGraw-Hill, second edition, 1997.

[3] Richard Montgomery, "A new solution to the three-body problem," *Notices of The AMS*, vol. 48, no. 5, pp. 471–481, May 2001.

[4] Frank Thomas Tveter, "Hamiltonian's equations of motion for non-conservative systems," *Celestial Mechanics and Dynamical Astronomy*, vol. 60, pp. 409–419, 1994.

[5] Stephan Bohacek and Edmond A. Jonckheere, "Linear dynamically varying LQ control of nonlinear systems over compact sets," *IEEE Transactions on Automatic Control*, vol. 46, no. 6, pp. 840–852, June 2001.

[6] Stephan Bohacek and Edmond Jonckheere, "Structural stability of linear dynamically varying (LDV) controllers," *Systems and Control Letters*, vol. 44, pp. 177–187, 2001.