

Indirect Control Invariance of Decoherence-Splitting Manifold (DSM)

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Abstract—Quantum coherence of open quantum systems is usually compromised because of the interaction with the ambient environment. Here, we introduce the concept of Decoherence Splitting Manifold (DSM), a submanifold of the space of density operators, wherein the system’s density matrix clearly exhibits a splitting between a unitarily evolving sub-density corresponding to some given set of its eigenvalues, which we aim to preserve, and another subdensity, in which eigenvalues are not preserved. The eigenspace of the eigenvalues that are preserved provides the concept of (time-varying) Decoherence Protected Subspace (DPS), a concept that subtly differs from the (time-varying) Decoherence Free Subspace (DFS) in the way the protected subspace is embedded in the full Hilbert space. With the DSM concept, the DPS is reformulated as a complex vector bundle over the DSM. In the absence of the traditional DFS, looking for less restrictive spaces and/or utilizing quantum control may help generate and/or retain a decoherence-free subevolution. This is achieved by searching for a DSM whose tangent bundle distribution is invariant under indirect control of an auxiliary qubit.

I. INTRODUCTION

In past few years, various methods have been proposed and implemented to mitigate the deleterious effect of decoherence in quantum computers and communication systems. For example, the notion of *Decoherence-Free Subspace (DFS)* was introduced as a passive method to bypass decoherence [13], [15], [21], [25]. In these methods, information is stored and processed in a protected subspace of the system Hilbert space, a subsystem, or a hybrid form of them.

States of a quantum system are represented by density matrices, Hermitian trace-1 positive semidefinite $n \times n$ matrices ϱ defined on the system Hilbert space $\mathcal{H} \cong \mathbb{C}^n$. In the absence of decoherence, evolution of a closed quantum system is described by a unitary transformation $\tilde{V}(t) \in U(n)$ such that $\varrho(t) = \tilde{V}(t)\varrho(0)\tilde{V}^*(t)$, whereas decoherence generally results in non-unitarity. A non-unitary transformation implies irreversibility of the dynamics, hence loss of information. However, certain *symmetries* of the system dynamics can yield a unitary sub-dynamics in some part of \mathcal{H} . Roughly speaking, a DFS is the subspace associated with such a sub-dynamics.

A *subspace* of \mathbb{C}^n is termed *Decoherence-Free* if its corresponding sub-dynamics is unitary. We represent the

decoherence-free state as $\varrho_{\text{DFS}} := \mathcal{P}_{\text{DFS}}\varrho\mathcal{P}_{\text{DFS}}$, where \mathcal{P}_{DFS} is the projector. If a state ϱ_{DFS} is decoherence-free, then for all times t there exists a unitary $\tilde{V}_K(t)$ such that

$$\varrho_{\text{DFS}}(t) = \tilde{V}_K(t)\varrho_{\text{DFS}}(0)\tilde{V}_K^*(t). \quad (1)$$

Essentially, we aim at developing relations of the form (1) using a novel procedure, certainly in the spirit of, but subtly different from, the decoherence-free concept, and renamed decoherence *protected* approach. Furthermore, in the traditional definition of DFS, it is further assumed that the projector \mathcal{P}_{DFS} is time-independent, meaning that the DFS is time-invariant. A precursor to the concept of time-varying DFSs has been introduced in [19], wherein unitarily-correctible subsystems are interpreted as time-varying noiseless spaces for open quantum systems. In the present work, the DPS depends on the time, but this does not mean that DPS is a mere time-varying DFS. In this work, \mathcal{P}_{DPS} is *fundamentally* time-dependent (more precisely $\varrho(t)$ -dependent), even for a time-invariant master equation. In fact, the subtle difference between DFS and DPS is probably best illustrated by the fact that, while DFS is a passive, *symmetry-induced* approach, DPS relies crucially on its *control-induced* aspect to achieve (1). The reliance on control can be regarded as a curse, but it has the blessing of universality, in the sense that the DPS is a matter of the topology of the set of densities, independently on the master equation. The DPS and the dynamics are then matched in a geometrically-inspired technique. While the early version [9] used traditional control-invariance, here, the approach is completely novel in that it relies on polynomial algebra.

II. DECOHERENCE PROTECTED CONCEPTS

A. Master equation set-up

Under some conditions [2], [4], the evolution of the system in its embedding environment can be described by the following master equation [in the units of $\hbar \equiv 1$]:

$$\dot{\varrho} = -i[H_0 + \sum_{\beta \neq 0} H_{\beta}u_{\beta}(t), \varrho] + \sum_{\gamma} L_{\gamma}(\varrho, u)w_{\gamma}(t). \quad (2)$$

Here, the Hermitian matrix H_0 denotes the free-evolution Hamiltonian of the open system (including some generically small Lamb shift corrections [2], [4]). The term $\sum_{\beta \neq 0} H_{\beta}u_{\beta}(t)$ is the time-varying *control* Hamiltonian, with real-valued “knobs” $u_{\beta}(t)$. The term $\sum_{\gamma} L_{\gamma}(\varrho, u(t))w_{\gamma}(t)$ encapsulates the interaction with the environment possibly depending on the control $u(t)$ and possibly involving time-varying decoherence rates $w_{\gamma}(t) \geq 0$, $\gamma = 1, 2, \dots$ to allow for complex system-reservoir interactions [5, Eq. (34)]. Here, instead of using

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the popular control-independent ‘‘Lindbladian,’’ we will consider more general models, where the decoherence explicitly depends on the system Hamiltonian, hence on the control $u(t)$. This includes the so-called *weak coupling limit* [4]. (In [4], the control-dependent Lindblad-Kossakowski term is written $\epsilon^2 L^\sharp(t)\varrho(t)$.) Even in the weak coupling limit, Eq. (2) is still an approximation, as it disregards terms depending on $\dot{u}(t)$ (written $K(t)\varrho(t)$ in [4]), which appear when the total Hamiltonian has time-varying eigenprojections (written $\Pi_j(t)$ in [4]). In the *singular coupling limit* on the other hand, the decoherence is so much faster than the drift that the dependency of the decoherence on the control is no longer present [4]. One of the purposes of this paper is to investigate the difference between a $L(\varrho)$ model and a $L(\varrho, u)$ model from the control perspective.

B. Desired unitary subdynamics

The fundamental concept is to keep one or several blocks of the eigenvalues of $\varrho(t)$ constant. As we shall see, this secures a unitary, decoherence-protected evolution along the corresponding eigenspace.

Consider the spectral decomposition of $\varrho(t)$,

$$\varrho(t) = \sum_{i=1}^n \lambda_i(t) |e_i(t)\rangle \langle e_i(t)|, \quad (3)$$

where $0 \leq \lambda_i(t) \leq 1$, $\sum_{i=1}^n \lambda_i(t) = 1$, $\lambda_i(t) \leq \lambda_j(t)$ for $i < j$, $|e_i(t)\rangle \in \mathcal{H}$, $\langle e_i(t)| \in \text{dual}(\mathcal{H})$, and $\langle e_i(t)|e_j(t)\rangle = \delta_{ij}$. Specifically, we assume the following multiplicity/degeneracy structure for the eigenvalues:

$$\lambda_{\sum_{i=1}^{k-1} m_i+1} = \lambda_{\sum_{i=1}^{k-1} m_i+2} = \dots = \lambda_{\sum_{i=1}^{k-1} m_i+m_k} \equiv \lambda_{[k]}.$$

We represent the above eigenvalues with the diagonal matrix $\Lambda_k = \lambda_{[k]} I_{m_k \times m_k}$, subject to $\sum_{k=1}^d m_k = n$ (d distinct blocks). Under this multiplicity condition and if $\varrho(t)$, as the solution to the master equation, is continuously differentiable, then the $|e_i(t)\rangle$'s can be taken as continuously differentiable orthonormal. This result is trivial when $m_1 = \dots = m_d = 1$, as the eigenvectors provide an orthonormal basis of \mathbb{C}^n . In the case of multiple eigenvalues, the eigenspace $E_k(t)$ of $\Lambda_k(t)$ *does* have a continuously differentiable orthonormal basis, because the base space, here the time-axis, is contractible. (This is the Atiyah-Dolezal theorem [11].) Since the E_k 's, $k = 1, \dots, d$, are mutually orthogonal, all bases of all eigenspaces E_k can be combined to provide a continuously differentiable orthonormal basis of \mathbb{C}^n made up with eigenvectors of $\varrho(t)$.

Now we assume that we want to unitarily preserve the blocks $\oplus_{k \in K} \Lambda_k =: \Lambda_K$ for some given subset $K \subseteq \{1, 2, \dots, d\}$ of the blocks, while the complementary eigenvalues $\oplus_{\bar{k} \in \bar{K}} \Lambda_{\bar{k}}(t) =: \Lambda_{\bar{K}}(t)$, $K \sqcup \bar{K} = \{1, 2, \dots, d\}$, are not numerically determined but determined in that their multiplicities remain unchanged. The subspace E_k corresponding to Λ_k can be uniquely identified with the eigenprojection

$$\mathcal{P}_k \equiv \sum_{l=1}^{m_k} \left| e_{\sum_{i=1}^{k-1} m_i+l} \right\rangle \left\langle e_{\sum_{i=1}^{k-1} m_i+l} \right|.$$

We now define the *protected subdensity* as follows:

$$\varrho_{\text{DPS}}(t) \equiv \mathcal{P}_{\text{DPS}}(t)\varrho(t)\mathcal{P}_{\text{DPS}}(t)/\text{Tr}[\mathcal{P}_{\text{DPS}}(t)\varrho(t)\mathcal{P}_{\text{DPS}}(t)], \quad (4)$$

in which $\mathcal{P}_{\text{DPS}}(t) \equiv \sum_{k \in K} \mathcal{P}_k(t)$.

It is easily seen that one can always find a continuously differentiable matrix $\tilde{V}_K(t)$ transforming the Λ_K -eigenvectors at $t = 0$ to those at $t > 0$; that is, $|e_{i \in I_K}(t)\rangle = \tilde{V}_K(t)|e_{i \in I_K}(0)\rangle$, where $I_K = \{i : \lambda_i \in \Lambda_K\}$. Hence the protected subdensity evolves unitarily:

$$\begin{aligned} \varrho_{\text{DPS}}(t) &= \sum_{i \in I_K} \lambda_i |e_i(t)\rangle \langle e_i(t)| / \sum_{i \in I_K} \lambda_i \\ &= \tilde{V}_K(t)\varrho_{\text{DPS}}(0)\tilde{V}_K^*(t). \end{aligned} \quad (5)$$

Further, the unitary subdynamics can be written in the usual Hamiltonian format:

$$\dot{\varrho}_{\text{DPS}}(t) = -i[H_{\text{EFF}}(t), \varrho_{\text{DPS}}(t)], \quad (6)$$

where

$$H_{\text{EFF}}(t) = i\dot{\tilde{V}}_K(t)\tilde{V}_K^*(t) \quad (7)$$

is the effective Hamiltonian.

C. Desired subdynamics in master equation solution

The desired unitary sub-dynamics was derived independently of H_0 and H_β . It thus remains to determine whether the desired unitary sub-dynamics is compatible with the equation of motion for $\varrho(t)$ [Eq. (2)], possibly after implementation of some control. From Eq. (4) we have

$$\dot{\varrho}_{\text{DPS}} = (\dot{\mathcal{P}}_{\text{DPS}}\varrho_{\text{DPS}} + \mathcal{P}_{\text{DPS}}\dot{\varrho}_{\text{DPS}} + \mathcal{P}_{\text{DPS}}\varrho_{\text{DPS}}\dot{\mathcal{P}}_{\text{DPS}}) / \sum_{i \in I_K} \lambda_i,$$

where we have used the fact that $\sum_{i \in I_K} \lambda_i$ is constant. From the definition of $\mathcal{P}_{\text{DPS}}(t)$ it is evident that

$$\dot{\mathcal{P}}_{\text{DPS}}(t) = -i[H_{\text{EFF}}(t), \mathcal{P}_{\text{DPS}}(t)].$$

After some algebra and taking Eq. (6) into account, we obtain

$$\mathcal{P}_{\text{DPS}}(t)\dot{\varrho}(t)\mathcal{P}_{\text{DPS}}(t) = -i\mathcal{P}_{\text{DPS}}(t)[H_{\text{EFF}}(t), \varrho(t)]\mathcal{P}_{\text{DPS}}(t). \quad (8)$$

One can simplify the above equation further and extract a relation including $\tilde{V}_K(t)$, $H_{\text{EFF}}(t)$, H_0 , $\{H_\beta\}$, $\{F_\gamma\}$, and $\varrho(0)$. Of course, here, we seek more general (time-varying) solutions. The issues are summarized in the following:

Problem 1: Given desired $H_{\text{EFF}}(t)$ and K , and specifying a model comprising of the Hamiltonian H_0 , a Lindbladian $L(\cdot)$, a control Hamiltonian set $\{H_\beta\}$, and $\varrho(0) = \varrho_0$, solve, if possible, Eq. (8) to find appropriate control knobs $u_\beta(t)$.

III. DECOHERENCE SPLITTING MANIFOLD

A. DSM concept

Problem 1 is easily seen to be equivalent to the following:

Problem 2: Find $u(\cdot)$ such that

$$\varrho(t) = V(t)\text{diag}\{\Lambda_K, \Lambda_{\bar{K}}(t)\}V^*(t),$$

where Λ_K is a fixed block of eigenvalues and $\Lambda_{\bar{K}}(t)$ is a block of possibly time-varying eigenvalues, but with constant multiplicities. ■

Since we are primarily interested in the eigenvalues of ϱ disregarding the eigenvectors, we introduce the following concept:

Definition 1: $\mathcal{D}_{\Lambda_K, m_{\bar{K}}}$ is the set of all densities of the form,

$$\varrho = V \text{diag} \{ \Lambda_K, \Lambda_{\bar{K}} \} V^*,$$

where Λ_K is fixed and $\Lambda_{\bar{K}}$ is a block of numerically unspecified eigenvalues but with fixed multiplicities $m_{\bar{K}}$.

Theorem 1: $\mathcal{D}_{\Lambda_K, m_{\bar{K}}}$ is a real-analytic manifold of real dimension $n^2 + \sum_{\bar{k} \in \bar{K}} m_{\bar{k}} - \sum_{\bar{k}} m_{\bar{k}}^2 - \sum_k m_k^2 - 1$.

Proof: See [9]. \blacksquare

$\mathcal{D}_{\Lambda_K, m_{\bar{K}}}$ is an example of a *Decoherence Splitting Manifold (DSM)*. The terminology of *splitting* manifold stems from the fact that in its tangent bundle there is a clear splitting between, on the one hand, a given subset of eigenvalues that are, with their multiplicities, preserved with the corresponding eigenvectors evolving unitarily and, on the other hand, a complementary subset of eigenvalues that are not preserved, but whose multiplicities are preserved.

B. DSM design: Universality

It is important to observe that the overall design is decoupled between (i) the DSM, which given the dimension of the relevant space of density operators depends only on how the eigenvalues are split between the preserved and the unpreserved ones, and (ii) the dynamics both in terms of the drift, the control Hamiltonian, and the decoherence process. The DSM (i) and the dynamics (ii) are matched by securing

$$\varrho(t) \in \text{DSM}, \quad \text{or} \quad \dot{\varphi}(t) \in T\text{DSM}.$$

From this point of view, the DSM is *universal* relative to all density evolution master equations.

IV. DECOHERENCE PROTECTED SUBSPACE

A. DPS as a vector bundle

In the above setup, the decoherence protected subspace is defined as the time-varying eigenspace of the preserved eigenvalues, $\text{DPS}(t) = \bigoplus_{k \in K} E_k(\varrho(t))$. Instead of “time-varying subspace,” DPS is more formally defined as a vector bundle over the DSM, as depicted in the following vector bundle diagram:

$$\begin{array}{ccc} \overbrace{\bigoplus_{k \in K} \mathbb{C}^{m_k}}^{\text{fiber}} & \rightarrow & \sqcup_{\varrho \in \text{DSM}} \overbrace{\bigoplus_{k \in K} E_k(\varrho)}^{\text{DPS}(\varrho)} \\ & & \downarrow \\ & & \text{DSM} = \mathcal{D}_{\Lambda_K, m_{\bar{K}}} \end{array}$$

Indeed, DPS acquires its time dependency exclusively through the motion $\varrho(t)$ in the base space.

B. Evolution of DPS on Grassmannian manifold

Consider the unitary evolution $\tilde{V}_K(t)$ of the eigenvectors associated with the constant eigenvalues Λ_K of the density operator. If we choose the computational basis $(0, \dots, 0, 1, 0, \dots, 0)^T$ for $|e_{i \in I_K}(0)\rangle$, $\tilde{V}_K(t)$ can be viewed as the $U(n)$ -matrix partitioned as $(\tilde{V}_{K,K}(t) \quad \tilde{V}_{K,\bar{K}}(t))$, where $\tilde{V}_{K,K}(t)$ denotes the matrix made up with the columns $|e_{i \in I_K}(t)\rangle$. However, the only specification on $\tilde{V}_K(t)$ is that it should map the orthonormal m_K -frame $|e_{i \in I_K}(0)\rangle$

to the orthonormal m_K -frame $|e_{i \in I_K}(t)\rangle$, regardless of the remaining $(n - m_K)$ -frame in the orthogonal complement. Thus, $\tilde{V}_K(t) \in U(n)/U(n - m_K)$. The latter is the Stiefel manifold $\mathbb{V}_{m_K}(\mathbb{C}^n)$ of m_K -frames in \mathbb{C}^n . With the preceding concepts, $\text{DPS}(t)$ is the column span of $\tilde{V}_{K,K}(t)$, as such $\text{DPS}(t) \in U(n)/U(m_K) \times U(n - m_K)$. The latter is the Grassmannian manifold $\mathbb{G}_{m_K}(\mathbb{C}^n)$ of m_K -dimensional complex subspaces of \mathbb{C}^n .

All of the above concepts are intertwined in the following fiber map, the principal bundle of the well-known universal bundle with $U(m_K)$ -structure group:

$$\begin{array}{ccc} U(m_k) & \xrightarrow{i} & U(n)/U(n - m_K) & \ni & \tilde{V}_K(t) \\ & & \downarrow \pi & & \downarrow \\ & & U(n)/U(m_K) \times U(n - m_K) & \ni & \text{DPS}(t) \end{array}$$

In the above, i is the inclusion and π the bundle projection. As is well-known, this bundle is far from trivial. Accordingly, we might not have a globally defined $\tilde{V}_K(t)$.

V. GEOMETRIC ASPECTS OF DECOHERENCE CONTROL

Decoherence control can be viewed as a disturbance rejection problem. Under its most traditional interpretation, the disturbance w is considered inaccessible [17, Problem 1, Theorem 1], possibly stochastically varying, as it happens under complex system-reservoir interaction [3]. The conventional geometric theory solution [7, Theorem 3.1] would define a controlled-invariant distribution Δ , containing the decoherence process distribution ($\Delta \supseteq \text{span} \{L\}$), and contained in (the tangent bundle of) some subspace in which the decoherence would not be too damaging for the given application. This subspace is undoubtedly the DSM, so that the traditional geometric condition for rejection of the decoherence would be $\text{span} \{L\} \subseteq \Delta \subseteq \Delta\Sigma M$, where $\Delta\Sigma M$ is the distribution that has DSM as integral manifold.

This is the point where our work departs from Ref. [6], which utilizes an output quantifying undesirable system-bath interactions, and opts not to utilize the density matrix as, quoted from [6, Sec. II], “*such tools are not the most convenient when analyzing the controllability and other basic geometric properties of open quantum systems.*” Ref. [22], however, develops an algebraic, albeit not geometric, invariance theory for density models.

Unlike the traditional geometric control condition for disturbance rejection, here we make $\Delta\Sigma M$ controlled invariant so that the traditional decoherence control condition simplifies to

$$\text{span} \{L\} \subseteq \Delta\Sigma M, \quad \Delta\Sigma M \text{ controlled-invariant.} \quad (9)$$

From the fundamental Bode paradigm, the only solution to the decoherence control problem, *under the assumption that w is not accessible*, is a *feedback* from $\varrho(t)$ to $u(t)$ around the point of entry of the uncertainty w in the block-diagram structure. The *inescapable* fact that feedback is necessary to reduce the effect of the uncertainty w should be pondered again the quantum mechanical measurement back-action, which unfortunately introduces a Lindbladian-like noise term, worsening the decoherence.

In order to, if not eliminate, at least reduce the reliance on feedback, one has to ask the question as to whether w is really unaccessible [5]. If w can be measured at least over some subspace, then this information allows some partial substitution of feedforward for feedback, as it has become customary in the context of self-bounded controlled-invariance [1], [17], [18].

In the present paper, we follow the reverse chronological order in which geometric control theory was developed (from control-invariance to self-bounded control invariance). We first consider a problem reminiscent of self-bounded control invariance: the case where all decoherence rates are known. In this case, the control law can at the same time annihilate the decoherence and make $\Delta\Sigma M$ controlled invariant. Under this condition, the problem simplifies to

$\Delta\Sigma M$ controlled-invariant.

From the above, clearly, we only indirectly go along the road of self-bounded control invariance. In addition, rather than utilizing subspace recursion of (self-bounded) control invariance as in [6], we develop a *novel* polynomial algebra technique to check the above.

VI. GEOMETRIC DSM DESIGN

A. Coordinate-free approach to DSM tangent bundle

Write the characteristic polynomial of the $n \times n$ density operator as

$$\chi_\varrho(s) = \sum_{q=0}^n \alpha_q(\varrho) s^{n-q}. \quad (10)$$

The polynomial is monic, hence, $\alpha_0 = 1$; furthermore, the density property implies that $\alpha_1 = -1$. To enforce the fact that the density matrix has $\lambda_1 = \dots = \lambda_{m_1}$ among its eigenvalues, we cancel the characteristic polynomial for $s = \lambda_1$ along with its derivatives up to order $(m_1 - 1)$. More generally, if the density operator has eigenvalues $\lambda_{m_1+\dots+m_{k-1}+1} = \dots = \lambda_{m_1+\dots+m_{k-1}+m_k}$ for $k \in K$, the conditions on the characteristic polynomial are

$$\begin{aligned} \chi_\varrho(\lambda_{m_1+\dots+m_{k-1}+m_k}) &= \chi_\varrho^{(1)}(\lambda_{m_1+\dots+m_{k-1}+m_k}) \\ &= \dots = \chi_\varrho^{(m_k-1)}(\lambda_{m_1+\dots+m_{k-1}+m_k}) = 0, \end{aligned}$$

where the superscript denotes the order of the derivative relative to s . Next if we want those eigenvalues to be preserved along an elementary motion $d\varrho$, we enforce the following conditions:

$$\begin{aligned} d\chi_\varrho(\lambda_{m_1+\dots+m_{k-1}+m_k}) &= d\chi_\varrho^{(1)}(\lambda_{m_1+\dots+m_{k-1}+m_k}) \\ &= \dots = d\chi_\varrho^{(m_k-1)}(\lambda_{m_1+\dots+m_{k-1}+m_k}) = 0, \end{aligned}$$

where d denotes the differential relative to the ϱ variables. As such, the $d\chi_\varrho^{(r)}(\lambda)$'s, for $r = 1, \dots, m_k - 1$, $\lambda = \lambda_{m_1+\dots+m_k}$, $k \in K$, are linear forms on the tangent bundle $T\text{Herm}(n)$. In order to make this clear, we rewrite the differentials as $d\chi_\varrho^{(r)}(\lambda)(\cdot)$. Define $d\chi_\varrho^{(*)}(\Lambda_K)(\cdot)$ to be the collection of all such linear forms.

Next, we need to enforce the multiplicity on the numerically unspecified eigenvalues. Take $\lambda_{[\bar{k}]}$ to be numerically unspecified but algebraically specified by its multiplicity, $m_{\bar{k}}$. As before we impose

$$\chi_\varrho(\lambda_{[\bar{k}]}) = \chi_\varrho^{(1)}(\lambda_{[\bar{k}]}) = \dots = \chi_\varrho^{(m_{\bar{k}}-1)}(\lambda_{[\bar{k}]}) = 0, \quad (11)$$

with the difference that the above should not hold for a prespecified eigenvalue, but that such eigenvalue must *exist*. Because the $\lambda_{[\bar{k}]}$'s are numerically unspecified, we need in addition to enforce

$$\sum_{k \in K} m_k \lambda_{[k]} + \sum_{\bar{k} \in \bar{K}} m_{\bar{k}} \lambda_{[\bar{k}]} = 1 \quad (12)$$

In other words, there must exist real $\Lambda_{\bar{K}}$ such that (11)-(12) hold. Therefore, we perform a Tarski-Seidenberg elimination of $\lambda_{[\bar{k}]}$, $\bar{k} \in \bar{K}$ among all polynomial equalities (11), differentiate, and add the resulting linear forms to $d\chi_\varrho^{(*)}(\Lambda_K)(\cdot)$ to obtain $d\chi_\varrho^{(*)}(\Lambda_K, m_{\bar{K}})(\cdot)$.

Using this formulation, the equation of the tangent bundle $T\mathcal{D}_{\Lambda_K, m_{\bar{K}}}$ becomes

$$d\chi_\varrho^{(*)}(\Lambda_K, m_{\bar{K}})(T\mathcal{D}_{\Lambda_K, m_{\bar{K}}}) = 0. \quad (13)$$

Lemma 1: The tangent bundle to the DSM can be extended to an involutive distribution $\Delta\Sigma M$.

Proof: By Theorem 1, under variation of one of the eigenvalues in the blocks Λ_K , a variation small enough so as to preserve the algebraic structure defined by the multiplicities, the various DSM's are real analytic manifolds, with their tangent spaces completing the original tangent space to a distribution. This distribution consists of the various tangent spaces to the various manifolds; hence it is integrable; hence by Frobenius's theorem it is involutive. ■

From the above, it follows that $\Delta\Sigma M$ can be characterized as

$$\Delta\Sigma M = \ker \left(d\chi_\varrho^{(*)}(\Lambda_K) \right). \quad (14)$$

B. Pauli basis DSM tangent bundle

We follow Ref. [20] and we define a state vector $\mathbf{x} \in \mathbb{R}^{n^2-1}$ whose components (in the standard computational basis) are $\varrho_{11} - \varrho_{ii}$, $i = 2, \dots, n$; $\varrho_{ij} + \varrho_{ji}$, $i > j$; $i\varrho_{ij} - \varrho_{ji}$, $i > j$. (Observe that the state $\mathbf{x} = \mathbf{0}$ has no coherence.) Then Eq. (2) can be rewritten in bilinear format:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x} + \sum_{\beta} (B_{\beta}\mathbf{x})u_{\beta} + \sum_{\gamma} (G_{\gamma}\mathbf{x})w_{\gamma}(t) \\ &= A\mathbf{x} + \mathcal{B}(I \otimes \mathbf{x})u + \mathcal{G}(I \otimes \mathbf{x})w. \end{aligned} \quad (15)$$

The matrices A , B_{β} , and G_{γ} are obtained as in Ref. [20] and $\mathcal{B} := \begin{pmatrix} B_1 & B_2 & \dots \end{pmatrix}$ and $\mathcal{G} := \begin{pmatrix} G_1 & G_2 & \dots \end{pmatrix}$.

If the collection $d\chi_\varrho^{(*)}(\Lambda_K, m_{\bar{K}})(\cdot)$ of forms is arranged in column-vector format, and if we switch to the \mathbf{x} -notation, we can define a Jacobian matrix from

$$d\chi_{\mathbf{x}}^{(*)}(\Lambda_K, m_{\bar{K}})(\cdot) = \left(J_{\chi_{\mathbf{x}}}^{(*)}(\Lambda_K, m_{\bar{K}}) \right) d\mathbf{x}(\cdot), \quad (16)$$

where we have recalled that the differentials $d\mathbf{x}$'s, arranged in column-vector format, are linear forms. Clearly, $\ker(J_{\mathbf{x}}) = T_x\text{DSM}$, where $J_{\mathbf{x}}$ is a shorthand for the Jacobian matrix.

C. Tangent bundle design

Write $a(\mathbf{x}) := (A + \mathcal{G} \cdot w)\mathbf{x}$, where $\mathcal{G} \cdot w := \sum_{\gamma} G_{\gamma} w_{\gamma}$, if w is known and constant or $a(\mathbf{x}) := A\mathbf{x}$ if w is unknown. Let $b(\mathbf{x})u := \mathcal{B}(I \otimes \mathbf{x})u$ where u is the control.

Theorem 2: Assume w is known and constant. If a control u^0 can be found such that $J_{\mathbf{x}}(a + bu^0) = 0$, this control makes $\Delta\Sigma M$ controlled-invariant.

Proof: Since $J_{\mathbf{x}}(a + bu^0) = 0$, we have $a + bu^0 \subseteq \ker(J_{\mathbf{x}}) = \Delta\Sigma M$. Let h be a distribution that completes $a + bu^0$ to a basis of $\Delta\Sigma M$. (This can always be done *locally*.) Thus $\text{span}\{a + bu^0, h\} = \Delta\Sigma M$. With these conventions, we have

$$\begin{aligned} [a + bu^0, \Delta\Sigma M] &= [a + bu^0, \text{span}\{a + bu^0, h\}] \\ &= [a + bu^0, a + bu^0] + [a + bu^0, h] \\ &= [a + bu^0, h] \\ &\subseteq \Delta\Sigma M \text{ (because } \Delta\Sigma M \text{ is involutive)}. \end{aligned}$$

Therefore, $[a + bu^0, \Delta\Sigma M] \subseteq \Delta\Sigma M$. \blacksquare

Observing that $J_{\mathbf{x}}(a + bu^0) = 0$ means that $J_{\mathbf{x}}\dot{\mathbf{x}} = 0$, Problem 1 can now be reformulated as follows:

Problem 3: Determine whether the equation

$$J_{\mathbf{x}}(A\mathbf{x} + \mathcal{B}(I \otimes \mathbf{x})u + \mathcal{G}(I \otimes \mathbf{x})w) = 0 \quad (17)$$

can be solved for u .

VII. FUNDAMENTAL DECOHERENCE CONTROL LIMITATIONS IN SINGULAR COUPLING CASE

The computational solution highlighted in Problem 3 exposes some fundamental limitations as to what simple actuator technology can achieve.

A. Lack of control authority with Hermitian control matrices

As we demonstrate here below, the classical *direct* control of spins by external magnetic fields yields $J_{\mathbf{x}}\mathcal{B}(I \otimes \mathbf{x}) = 0, \forall \mathbf{x}$. Precisely and more generally,

Theorem 3: For any hermitian control matrix H_{β} (which includes S_{ω}^i) and matrix B_{β} such that $B_{\beta}\mathbf{x} = -i[H_{\beta}, \varrho]$, where \mathbf{x} is the coordinate vector of ϱ in the Pauli basis, and for any DSM with tangent bundle given by the kernel of the Jacobian $J_{\mathbf{x}}$, we have $J_{\mathbf{x}}(\Lambda_K)B_{\beta}\mathbf{x} = 0, \forall \mathbf{x} \in \text{DSM}, \forall \Lambda_K$.

Proof: The master equation $\dot{\varrho}(t) = -i[H_{\beta}u_{\beta}(t), \varrho]$ induces unitary evolution on $\varrho(t)$, whatever the control $u_{\beta}(t)$:

$$\varrho(t) = e^{-iH_{\beta} \int_0^t u_{\beta}(\tau) d\tau} \varrho(0) e^{iH_{\beta} \int_0^t u_{\beta}(\tau) d\tau}.$$

The latter in turn means that all eigenvalues of $\varrho(t)$ are preserved, along with their multiplicities. Thus the trajectory $\varrho(t)$ is in a $\mathcal{D}_{\Lambda_K, \emptyset}$ that preserves *all* eigenvalues. In particular, some of the eigenvalues of Λ_K are preserved in the DSM; call Λ_{K_1} those eigenvalues; thus $\text{DSM} = \mathcal{D}_{\Lambda_{K_1}, m_{K_1}}$. Clearly, $\dot{\varrho} \in T\mathcal{D}_{\Lambda_K, \emptyset}$. But since $\mathcal{D}_{\Lambda_K, \emptyset} \subseteq \text{DSM}$, we have $\dot{\varrho} \in T\text{DSM}$. This is exactly what $J_{\mathbf{x}}B_{\beta}\mathbf{x}u_{\beta} = 0$ means. \blacksquare

Because of the above failure, the next idea would be to enlarge the set of admissible controls to include pulsing sequences generating controls in the Lie algebra $\mathcal{L}(\mathcal{B}(I \otimes x))$

of all vector fields $B_{\beta}x$. Contrary to [23], here, this effort is bound to fail, as it is trivially observed that

$$J_{\mathbf{x}}\mathcal{L}(\mathcal{B}(I \otimes \mathbf{x})) = 0.$$

B. Control ‘‘authority’’ with measurement back-action

Consider the master equation (2) augmented by the continuous observation equation [8]:

$$\begin{aligned} \dot{\varrho} &= -i[H_0 + \sum_{\beta \neq 0} H_{\beta}u_{\beta}(t), \varrho] - k[M, [M, \varrho]] \\ &\quad + \sum_{\gamma} L_{\gamma}(\varrho)w_{\gamma}(t) \\ &\quad + \sqrt{2k}(M\varrho + \varrho M - 2\langle M \rangle \varrho) \frac{dW}{dt}. \end{aligned} \quad (18)$$

The first line contains the free dynamics, the Hamiltonian control, and a back-action term $-k[M, [M, \varrho]]$ due to the measurement, where M is the measurement operator and $k \geq 0$ the strength of the measurement. The second line has the original decoherence terms involving the Lindbladians L_{γ} . The third line is a measurement-induced noise, where W is a Wiener process. We could think of using the back-action term k as a less classical ‘‘measurement-induced’’ control, and indeed, as the example of [10] shows,

$$d\chi_{\varrho}^{(*)}(\Lambda_K)([M, [M, \varrho]]) \neq 0.$$

If we define the mapping B_{μ} by $B_{\mu}(M)\mathbf{x} = ([M, [M, \varrho]])$, the above can be rewritten $J_{\mathbf{x}}B_{\mu}(M)\mathbf{x} \neq 0$. The preceding apparently allows for *some* authority, but because $k \geq 0$ this ‘‘control’’ is actually another decoherence term, making things worse.

However, it is well known that augmenting the system, putting a Hamiltonian control on the auxiliary system, and then tracing out the auxiliary system yields a master equation with another Lindbladian.

VIII. INDIRECT CONTROL BY SYSTEM AUGMENTATION

Consider two system qubits 1,2 along with an auxiliary qubit A , with coupling Hamiltonian $Z_1Z_A + Z_2Z_A + \sigma_A u$, where Z_{α} is the usual z -Pauli operator on spin α and σ_A is any of the x, y, z -Pauli operators on qubit A . The evolution of the 1-2 system is given by

$$\varrho(t) = \text{Tr}_A(e^{-iHt}(\varrho(0) \otimes \varrho_A(0))e^{iHt}).$$

By the Cayley-Hamilton theorem, there exists an interpolation polynomial $p_{t,u}(s)$ such that $p_{t,u}(H) = e^{-iHt}$. Upon normalizing u to be integer (pulsing control sequence), $p_{t,u}(H)$ is a trigonometric polynomial in e^{-it} . Furthermore, the system-auxiliary tensor product carries over to e^{-iHt} , the partial tracing is easily done, and $\varrho(t)$ is obtained. The differentiation $(d/dt)\varrho(t)$ is also easily done symbolically. Therefore, $\varrho(t)$ and $\dot{\varrho}(t)$ are trigonometric polynomials in e^{-it} . Assigning numerical values to the initial conditions (leaving them symbolically seems hard at this stage), and eliminating e^{-it} using a Tarski-Seidenberg procedure yields a model of the form $\dot{\mathbf{x}} = B^{x,y}(\mathbf{x})u$, for $\sigma_3 = X_3$ or Y_3 , while taking $\sigma_3 = Z_3$ does not lead to any control authority. With this choice, $J_{\mathbf{x}} \begin{pmatrix} B^x(\mathbf{x}) & B^y(\mathbf{x}) \end{pmatrix}$ is nonsingular.

IX. GEOMETRICALLY-INSPIRED DECOHERENCE CONTROL

A. Accessible decoherence rates

We first look at the singular coupling case, with the understanding that \mathcal{B} has some control authority acquired, for example, by system augmentation.

The number of equations to be satisfied in (17) equals the number of eigenvalues to be preserved (multiplicities counted) plus the number of constraints imposed by the constancy of the multiplicities $m_{\bar{K}}$; generically we need as many control inputs. The matrix $J_{\mathbf{x}}\mathcal{B}(I \otimes \mathbf{x})$ can be made “fat” with enough controls, and the solution to (17) is given by

$$u^0 = -(J_{\mathbf{x}}\mathcal{B}(I \otimes \mathbf{x}))^{-R} J_{\mathbf{x}}(A\mathbf{x} + \mathcal{G}(I \otimes \mathbf{x})w),$$

provided a right inverse $(J_{\mathbf{x}}\mathcal{B}(I \otimes \mathbf{x}))^{-R}$ of $J_{\mathbf{x}}\mathcal{B}(I \otimes \mathbf{x})$ exists.

Now, consider the weak limit case, where the Lindblad jump operators are control-dependent. Clearly, the dependency of \mathcal{G} on u is a *blessing* rather than a curse, as it provides more coupling between, on the one hand, the control and, on the other hand, the decoherence process. Clearly, the solvability of (17) is a matter of the inverse function theorem: a local solution exists iff the Jacobian (relative to the control)

$$\begin{aligned} J_u(J_{\mathbf{x}}(\mathcal{B}(I \otimes \mathbf{x})u + \mathcal{G}(u)(I \otimes \mathbf{x}))) \\ = J_{\mathbf{x}}(\mathcal{B}(I \otimes \mathbf{x})) + J_u(J_{\mathbf{x}}(\mathcal{G}(u)(I \otimes \mathbf{x}))) \end{aligned}$$

is right-invertible. Clearly, this is much more likely to happen because the additional coupling term $J_u(J_{\mathbf{x}}(\mathcal{G}(u)(I \otimes \mathbf{x})))$.

B. Inaccessible decoherence rates

If $w(t)$ is completely unknown, then the only way to make (17) hold is to impose

$$J_{\mathbf{x}}(\mathcal{G}(u)(I \otimes \mathbf{x})) = 0,$$

which is only possible if \mathcal{G} depends on u . Assuming that equation can be solved for u , the remaining condition is

$$J_{\mathbf{x}}(A\mathbf{x} + \mathcal{B}(I \otimes \mathbf{x})u) = 0.$$

Thus the design amounts to existence of a u satisfying the above two equations.

X. CONCLUSION

The bigger picture is two-fold: (i) identify how much and where information can be protected and once this is identified as a DSM (ii) find a control law that locks on the DSM. In case the decoherence rates are available, the restriction on (i) is not as much theoretical as it is a matter of the technology needed to achieve the control authority. In case of unknown decoherence rates, the present work is more relevant to (ii) than to (i).

The solution to (i) has here been left to a “trial and error” procedure to identify a distribution that is likely to contain the decoherence process. Another way to tackle the latter problem is via the more recent self-bounded control invariance: namely, find the smallest self-bounded control

invariant distribution Σ_*^L containing the decoherence process and find a DSM such that $\Sigma_*^L \subseteq \Delta\Sigma M$. This is left for further research.

APPENDIX

A. Review of Decoherence-Free concepts

The classical DFS approach conjectures existence of an orthogonal projection $P_{\text{DFS}} : \mathcal{H} \rightarrow \mathcal{H}_{\text{DFS}}$ such that the corresponding sub-density

$$\varrho_{\text{DFS}}(t) = \frac{1}{\text{Tr}(P_{\text{DFS}}\varrho(t)P_{\text{DFS}})} P_{\text{DFS}}\varrho(t)P_{\text{DFS}} \quad (19)$$

has a decoherence-free evolution

$$\dot{\varrho}_{\text{DFS}}(t) = -\imath[H_{\text{DFS}}, \varrho_{\text{DFS}}(t)] \quad (20)$$

relative to the reduced Hamiltonian

$$\mathcal{H}_{\text{DFS}} = P_{\text{DFS}}HP_{\text{DFS}}. \quad (21)$$

Eq. (20) guarantees unitary evolution

$$\varrho_{\text{DFS}}(t) = e^{-\imath H_{\text{DFS}}t} \varrho_{\text{DFS}}(0) e^{\imath H_{\text{DFS}}t} \quad (22)$$

To check consistency of (20) with the Lindblad master equation, we differentiate (19), make use of (20), to get

$$P_{\text{DFS}}(\dot{\varrho}(t) + \imath[H_{\text{DFS}}, \varrho(t)])P_{\text{DFS}} = 0 \quad (23)$$

This condition can be secured by the traditional DFS approach of choosing \mathcal{H}_{DFS} to be the span of common eigenvectors to the quantum jumps operators F_{γ} (see [15], [21]).

It is important to observe the *subtle point* that only those eigenvalues of the projection on $\varrho(t)$ on \mathcal{H}_{DFS} are preserved and not, in general, some subset of eigenvalues of $\varrho(t)$! The counterexample of Sec. C will serve to illustrate that point.

B. New Decoherence-Protected Concepts

To highlight the DFS versus DPS distinction, we first review the DPS scheme in a setting much simpler than the formal one of Section II-B.

Our newer scheme begins by conjecturing an evolution of the density solution to (2) that has the form

$$\varrho(t) = V(t) \text{diag} \{ \Lambda_K \quad \Lambda_{\bar{K}}(t) \} V^*(t), \quad (24)$$

where the eigenvalues in Λ_K are preserved, while the remaining eigenvalues in $\Lambda_{\bar{K}}(t)$ could be evolving, with the only restriction that there are no eigenvalue crossings. A density evolving as (24) will be said to evolve on a Decoherence Splitting Manifold (DSM). Such a solution might not exist through the natural drift dynamics H_0 , but it can be enforced by geometrically-inspired control.

Let $V_K(t)$ be the matrix of eigenvectors of Λ_K . If we define the decoherence protected sub-density

$$\varrho_{\text{DPS}}(t) \quad (25)$$

$$= \frac{1}{\text{Tr}(\Lambda_K)} V_K(t) \Lambda_K V_K^*(t) \quad (26)$$

$$= \frac{1}{\text{Tr}(\mathcal{P}_{\text{DPS}}(t)\varrho(t)\mathcal{P}_{\text{DPS}}(t))} \mathcal{P}_{\text{DPS}}(t)\varrho(t)\mathcal{P}_{\text{DPS}}(t),$$

where $P_{\text{DPS}} = V_K V_K^* : \mathcal{H} \rightarrow \mathcal{H}_{\text{DPS}}$, it is immediately obvious that this subdensity has unitary evolution:

$$\varrho_{\text{DPS}}(t) = V_K(t) V_K^*(0) \varrho_{\text{DPS}}(0) V_K(0) V_K^*(t) \quad (27)$$

If we differentiate (25), we get

$$\dot{\varrho}_{\text{DPS}}(t) = -i[H_{\text{EFF}}(t), \varrho_{\text{DPS}}(t)] \quad (28)$$

where

$$H_{\text{EFF}}(t) := -iV_K(t)\dot{V}_K^*(t) = i\dot{V}_K(t)V_K^*(t) \quad (29)$$

is an effective Hamiltonian.

To secure compatibility between (25) and the master equation, a bit of matrix analysis (see Sec. II-B for details) yields

$$\mathcal{P}_{\text{DPS}}(t) (\dot{\varrho}(t) + i[H_{\text{EFF}}(t), \varrho(t)]) \mathcal{P}_{\text{DPS}}(t) = 0. \quad (30)$$

The resemblance between (19)-(20)-(22)-(23) and (25)-(28)-(27)-(30) is striking, but it hides the important difference that in the former scheme eigenvalues of some projection of ϱ are preserved whereas in the latter some eigenvalues of ϱ are preserved. More specifically, in the DPS scheme, P_{DPS} is the projection onto an eigenspace of the density, whereas that need not be the case in the DFS case. In fact, even in the absence of drift dynamics, the most elementary quantum jump operators result in a P_{DFS} that is completely remote from the eigenspace of ϱ and as such *none* of the eigenvalues of ϱ are preserved under the DFS. The DFS-DPS difference will be made clear by the counterexample of the following section.

Another way to see the discrepancy is to observe that, in the DFS case, H_{DFS} is the projection of the Hamiltonian on the DFS subspace, while, in the DPS case, the reduced Hamiltonian H_{EFF} is just an ‘‘effective’’ Hamiltonian.

Because of the subtle difference in subspace in which the density is protected, we refer to the new subspace as *Decoherence Protected Subspace (DPS)*, as opposed to the classical Decoherence Free Subspace (DFS).

C. Counterexample

Consider a 2-qubit system without internal dynamics and subject to the simplest decoherence scheme:

$$\dot{\varrho}(t) = 2F\varrho(t)F^* - F^*F\varrho(t) - \varrho(t)F^*F, \quad (31)$$

where the quantum jump operator is $F = \sigma_z \otimes I + I \otimes \sigma_z$. We focus on the classical

$$\mathcal{H}_{\text{DFS}} = \text{span}\{|01\rangle, |10\rangle\},$$

where by convention $|0\rangle = (10)^T$ and $|01\rangle = |0\rangle \otimes |1\rangle$, $|10\rangle = |1\rangle \otimes |0\rangle$. With this convention,

$$\mathcal{H}_{\text{DFS}} = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{C}^4 =: \mathcal{H}.$$

Define the projection

$$P_{\text{DFS}} = \left(\begin{array}{c|c|c} 0 & 0_{[2:3],1} & 0 \\ \hline 0_{[2:3],1} & I_{[2:3],[2:3]} & 0_{[2:3],4} \\ \hline 0 & 0_{4,[2:3]} & 0 \end{array} \right)$$

along with (even though somewhat pedantic for this particular example) $H_{\text{DFS}} := P_{\text{DFS}} H P_{\text{DFS}}$ and $\varrho_{\text{DFS}}(t) = P_{\text{DFS}} \varrho(t) P_{\text{DFS}} / \text{Tr}(P_{\text{DFS}} \varrho(t) P_{\text{DFS}})$. All of the above yields the unitary evolution $\dot{\varrho}_{\text{DFS}}(t) = -i[H_{\text{DFS}}, \varrho_{\text{DFS}}(t)]$.

If we integrate the master equation (31), we find, using MATLAB notation,

$$\varrho(t) = \left(\begin{array}{c|c|c} \varrho_{11}(0) & \varrho_{1,[2:3]}(0)e^{-4t} & \varrho_{14}(0)e^{-16t} \\ \hline \varrho_{[2:3],1}(0)e^{-4t} & \varrho_{[2:3],[2:3]}(0) & \varrho_{[2:3],4}(0)e^{-4t} \\ \hline \varrho_{41}(0)e^{-16t} & \varrho_{4,[2:3]}(0)e^{-4t} & \varrho_{44}(0) \end{array} \right).$$

Clearly, consistently with the DFS, the information in the middle block is protected. The problem is that, contrary to our scheme, there are no preserved eigenvalues. For example, if

$$\varrho(0) = \left(\begin{array}{c|c|c|c} 0.2251 & 0.2037 & 0.1362 & -0.0100 \\ \hline 0.2037 & 0.3250 & 0.1125 & 0.1125 \\ \hline 0.1362 & 0.1125 & 0.1750 & 0.1000 \\ \hline -0.0100 & 0.1125 & 0.1000 & 0.2750 \end{array} \right),$$

we have

$$\lambda(\varrho(0)) = \{0.0036, 0.1195, 0.2732, 0.6037\},$$

whereas for $t = 0.25$, we have

$$\varrho(0.25) = \left(\begin{array}{c|c|c|c} 0.2251 & 0.0749 & 0.0501 & -0.0002 \\ \hline 0.0749 & 0.3250 & 0.1125 & 0.0414 \\ \hline 0.0501 & 0.1125 & 0.1750 & 0.0368 \\ \hline -0.0002 & 0.0414 & 0.0368 & 0.2750 \end{array} \right)$$

and

$$\lambda(\varrho(0.25)) = \{0.1124, 0.1809, 0.2665, 0.4403\},$$

and clearly none of the eigenvalues are preserved!

D. DPS versus time-varying DFS

In [24], the authors consider time-varying quantum jump operators $F_\gamma(t)$ in the case where the F_γ 's have common eigenvectors:

$$F_\gamma(t) |\Phi_i(t)\rangle = c_\gamma(t) |\Phi_i(t)\rangle, \quad \forall \gamma, \quad i = 1, 2, \dots$$

Then they construct the time-varying Decoherence-Free Subspace (t-DFS)

$$\mathcal{H}_{\text{t-DFS}}(t) = \text{span}\{|\Phi_i(t)\rangle : i = 1, 2, \dots\}.$$

The above is clearly an extension of a classical method for constructing the time-invariant DFS (see [14], [21], [16], [12], [15]).

The starting point of this time-varying extension certainly differs from our method, as $\mathcal{H}_{\text{t-DFS}}$ is defined in terms of the eigenspace of the quantum jump operators, while in our method it is defined from the eigenspace of ϱ . However, our method forces ϱ to lock on some DSM and the question is whether this could *indirectly* confer ϱ the eigenstructure of the quantum jump operators.

The authors of [24] make $\mathcal{H}_{t\text{-DFS}}$ invariant under an effective Hamiltonian

$$\begin{aligned} \bar{H}_{\text{eff}}(t) &= i\dot{U}(t)U^*(t) \\ &+ U(t) \left(H(t) + \frac{i}{2} \sum_{\gamma} (c_{\gamma}^*(t)F_{\gamma}(t) - c_{\gamma}(t)F_{\gamma}^*(t)) \right) U^*(t), \end{aligned} \quad (32)$$

where $H(t)$ is the original system Hamiltonian and

$$U(t) := \sum_{i=1}^K |\Phi_i(0)\rangle \langle \Phi_i(t)| + \sum_{j=K+1}^N |\Phi_j^{\perp}(0)\rangle \langle \Phi_j^{\perp}(t)|.$$

This effective Hamiltonian plays the same role as our H_{EFF} as

$$\frac{d}{dt} \bar{\varrho}(t) = -i [\bar{H}_{\text{eff}}(t), \bar{\varrho}(t)], \quad (33)$$

where

$$\bar{\varrho}(t) = U(t) \underbrace{\sum_i p_i |\varphi_i(t)\rangle \langle \varphi_i(t)|}_{\varrho(t)} U^*(t) \quad (34)$$

and $\varphi_i(t) \in \mathcal{H}_{t\text{-DFS}}(t)$. (Here we have slightly generalized [24] to allow for less trivial case than rank 1 densities.)

Let us compare (32)-(33)-(34) with (29)-(28)-(25). Eq. (34) is clearly the projection of the density on $\mathcal{H}_{t\text{-DFS}}(0) = \text{span}\{|\Phi_{\alpha}(0)\rangle : \alpha = 1, 2, \dots\}$ and *not* on the eigenspace of $\varrho(t)$ as in (25). Contrary to (29), the Hamiltonian (32) is not reduced, although using its invariance property,

$$\langle \Phi_i^{\perp}(0) | \bar{H}_{\text{eff}}(t) | \Phi_j(0) \rangle = 0,$$

it could be reduced and plugged into (33).

Let us now compare (33) and (28). The former means that some density preserves all of its eigenvalues, while the latter means that some projection of the density preserve its eigenvalues. To reconcile the two concepts, ϱ as defined by (34) must be the projection of some augmented density such that the eigenvalues of its projection are preserved. Make the φ_i 's in (34) orthonormal and complete ϱ as follows:

$$\begin{aligned} \varrho^{\text{augmented}}(t) &= \frac{1}{\sum_i p_i + \sum_j p_j^{\perp}(t)} \\ &\left(\sum_i p_i |\varphi_i\rangle \langle \varphi_i| + \sum_j p_j^{\perp}(t) |\varphi_j^{\perp}\rangle \langle \varphi_j^{\perp}| \right), \end{aligned}$$

where the $\varphi_j^{\perp}(t)$'s are also orthonormal and do not cross. Then the p_i 's and the $p_j^{\perp}(t)$'s are eigenvalues; and further the p_i 's are preserved. Hence $\varrho^{\text{augmented}}(t)$ evolves in a DSM. This similarity, however, hides the fact that our method is believed to be more general, to be able to track other protected subspaces than $\mathcal{H}_{t\text{-DFS}}$, as it uses control to broaden the range of solutions and does not rely on the classical structure of the decoherence term.

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