# Dirichlet's Principle on Multiclass Multihop Wireless Networks: Minimum Cost Routing Subject to Stability

Reza Banirazi, Edmond Jonckheere, Bhaskar Krishnamachari
Department of Electrical Engineering, University of Southern California, Los Angeles, CA 90089

## ABSTRACT

Minimum cost routing is considered on multiclass multihop wireless networks influenced by stochastic arrivals, inter-channel interference, and time-varying topology. Endowing each wireless link with a cost factor, possibly time-varying and different for each class, we define the Dirichlet routing cost as the square of link packet transmissions weighted by the link cost-factors. A dynamic routing protocol is proposed to minimize this cost, while ensuring queue stability for all stabilizable traffic demands, and without requiring any information about network topology or packet arrivals. Further, when all links are of unit cost factor, the proposed protocol leads to minimum average network delay among all routing protocols that act based only on current queue congestion and current channel states. We also show that under our proposed routing protocol, the fluid limit of wireless network follows Ohm's law on a suitably-defined nonlinear resistive network. This makes possible the use of tools from circuit theory, such as nonlinear resistive networks, to analyze and optimize the rate behavior of stochastic interfering wireless networks.

## 1. INTRODUCTION

Consider a time-slotted wireless network in which new packets of the same size may randomly arrive to different nodes, destined for any other node, potentially several hops away. Due to environmental factors and user mobility, the topology of the network may randomly change in time. Due to inter-channel interference, not all wireless channels can transmit at the same time. The network is described by a *simple, directed* connectivity graph with set of nodes $\mathcal{V}$ and directed edges $\mathcal{E}$. Packets of the same destination form a *class* (regardless of their sources) and $\mathcal{K}\subseteq\mathcal{V}$ represents the set of all possible classes in the network. Given a wireless link $ij \in \mathcal{E}$ and a class $d \in \mathcal{K}$, the link *actual-transmission* $f_{ij}^{(d)}(n)$ counts the number of $d$-class packets (or $d$-packets in short) transmitted over the link at timeslot $n$. For each class $d$, every link $ij$ is also endowed with a link *cost-factor* $\rho_{ij}^{(d)}(n) \geqslant 1$ that represents the cost of transmitting one $d$-packet over the link at slot $n$. The aim of this paper is to find time-slotted actual-transmissions $f_{ij}^{(d)}(n)$ that solve

$$\text{Minimize: } \limsup_{\tau\to\infty} \frac{1}{\tau} \sum_{n=0}^{\tau-1} \mathbb{E}\Big\{ \sum_{ij\in\mathcal{E}} \sum_{d\in\mathcal{K}} \rho_{ij}^{(d)}(n) \big(f_{ij}^{(d)}(n)\big)^2 \Big\} \quad (1)$$

Subject to: Queue stability for all stabilizable arrivals

where $\mathbb{E}$ denotes expectation. We refer to the cost function of problem (1) as *Dirichlet routing cost* due to its connection with Dirichlet's Principle in mathematics [1].

It is worth noting that the problem (1) needs to be solved at *network and link layer,* which totally disconnects the treatment of this problem from cross-layer optimization techniques [2–5]. The aim in the latter is to control flow at transport layer so as to keep the arrival rates within the network capacity region. The network-layer routing policy, on the other hand, has no control on arrivals. Instead, the basic assumption here is queue stabilizability, meaning that the arrival rates lie within the network capacity region. Then, an important quality factor of a protocol, called *throughput optimality,* is to stably support the entire capacity region. It should be obvious that minimum cost routing at network and link layer has no contradiction with flow control at transport layer.

When the link cost-factors are correlated with a sort of resource consumption such as energy (resp., quality defect such as end-to-end delay), the constrained minimization (1) is closely related to minimizing average network resource (resp., average quality defect) subject to throughput optimality. The quadratic cost function of (1) can represent different routing penalties, giving a wide-reaching impact to the optimization problem (1). Below are some examples that can be brought into this abstract model.

- *Link quality factor turned as a cost factor:* The quality of wireless communication depends on hardware and environmental factors. Different state-of-the-art link-quality metrics have been introduced such as Expected Transmission Count (ETX), Packet Reception Rate (PRR), Required Number of Packet Transmissions (RNP), etc. (See [6] for a review.) Most link quality measures can easily be merged with problem (1) as a link cost-factor.
- *Routing distance minimization:* The cost factor of each link $ij$ for a class $d$ can be assigned proportional to the hop-count or geographic distance between the receiving node $j$ and the destination node $d$. Then, due to the close correlation between routing distance and packet latency, problem (1) may be interpreted as average delay minimization subject to throughput optimality.
- *Energy usage minimization:* Link capacity is a function of transmission power and channel condition. Assuming that power allocation happens independently of queue congestion, each link may receive a cost factor proportional to the ratio of allocated power to the link capacity.

• *Relaying cost minimization:* Nodes in ad-hoc networks, which belong to different users, may act in their own interests rather than forwarding traffic for others. Several protocols have addressed this non-cooperative issue, based mostly on game theory and with no cost optimization or throughput optimality result (see [7] and references therein). Assume that every node declares a relaying cost to transfer one packet. Then in our framework, each incoming link to a node may receive a cost factor proportional to the node relaying cost.

**Related Work.** It is shown in [8] that a stationary randomized algorithm can solve the constrained optimization problem (1). While such an algorithm exists in theory, it is intractable in practice as it requires a full knowledge of arrival statistics and channel state probabilities. Moreover, assuming all the statistics and probabilities could be accurately estimated, the algorithm would still need to solve a dynamic programming problem for each topology state, where the number of states grows exponentially with the number of wireless channels.

Thus far, the V-parameter Back-Pressure (BP) has been the only feasible approach to decreasing, but not minimizing, a routing penalty at network layer [8]. It parametrizes the original BP [9] by a constant $V \geqslant 0$ to trade queue congestion for routing cost. It is proved that the average routing cost can come within $O(1/V)$ of its minimum, but at the expense of an increase in average network delay of $O(V)$ relative to that of the original BP. Therefore, while the algorithm can decrease a routing cost towards its minimum, it is not able to achieve the minimum cost subject to network stability. We remark, however, that the V-parameter approach holds for more general cost functions, and is not restricted to the particular structure of the Dirichlet routing cost in (1).

Adding distance information to link weights, [10] enhances BP to give priority to shorter paths. It does not minimize the average routing distance and uses the shortest path information in a heuristic manner. Energy-delay tradeoff for sensor networks was considered in [11]. Asymptotic energy usage as network size grows to infinity was studied in [12]. The V-parameter BP was used in [13] to reduce average energy usage for a multihop wireless network. A better energy-delay tradeoff was introduced in [14] for the special case of wireless downlinks. The work in [15] used ETX metric with V-parameter BP to decrease average packet transmissions. The authors in [16] restricted BP to the number of hop-counts on each flow, assuming that each node knows a-priori its hop distance from all others. Using greedy embedding, [17] modified BP scheduling by giving priority to the links with a shorter hyperbolic distance to the final destination.

This paper follows our recent research on developing a wireless routing protocol inspired by heat diffusion process [18,19]. Specifically, here we extend the results of [18] on the Dirichlet routing cost in two directions: First, rather than taking a free-capacity directed graph as in [18], we formalize Dirichlet's Principle on a capacity-constrained directed graph as the genuine representation of a data network. Second, we consider multiclass networks that together with limited edge capacities make Dirichlet's Principle far more complicated and unorthodox.

**Contribution.** First, we propose a throughput-optimal routing protocol that solves the cost minimization problem (1) without requiring any information about network topology or packet arrivals. This is the first time in literature that a viable network-layer protocol can minimize a general routing cost while ensuring queue stability.

Second, when all links are of unit cost factor, the proposed protocol leads to minimum time average total queue congestion in the network, which is proportional to average network delay by Little's Theorem.

Third, under our routing protocol, the long-term average dynamics (fluid limits) of the wireless network comply with Ohm's law. This opens a way to take advantage of tools from circuit theory, such as nonlinear resistive networks, in the analysis and optimization of stochastic packet networks under link interference.

Fourth, our routing protocol enjoys the same algorithmic structure, complexity, and overhead signaling as BP. Thus all advanced improvements to BP (see [8] and references therein) can easily be leveraged to further enhance our protocol too. This also simplifies the software transition to practice from existing BP implementations.

**Notation.** For $\mathcal{S}$ as a set, $|\mathcal{S}|$ denotes its cardinality. A superscript $\top$ denotes the transpose operation. For $\boldsymbol{v}$ a vector, $\|\boldsymbol{v}\| := (\boldsymbol{v}^\top \boldsymbol{v})^{1/2}$ denotes its norm. For $\boldsymbol{v}$ as a block vector, $\mathrm{diag}(\boldsymbol{v})$ denotes its block diagonal matrix expansion. We denote the zero vector with $\mathbf{0}$, the vector of all ones of size $m$ with $\mathbf{1}_m$, and the identity matrix of size $m$ with $\boldsymbol{I}_m$. Between vectors or matrices, the entrywise (Schur) product is denoted with $\odot$ and the tensor product with $\otimes$. For two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, the operators $\min\{\boldsymbol{u}, \boldsymbol{v}\}$ and $\max\{\boldsymbol{u}, \boldsymbol{v}\}$, also curly inequalities $\preccurlyeq$ and $\succcurlyeq$, act entrywise. For $x$ as a real number, $\lfloor x \rfloor$ maps $x$ to the largest preceding integer, and $\lceil x \rceil$ to the smallest following integer. The indicator function $\mathbb{I}_X$ takes the value 1 if the statement $X$ is true, and 0 otherwise. For a vector $\boldsymbol{v}$, we define $\boldsymbol{v}^+ := \max\{\mathbf{0}, \boldsymbol{v}\}$. On a graph, or a network, for a value $x$ corresponding to a directed edge $\ell$ from node $i$ to node $j$, we use the notation $x_\ell$ and $x_{ij}$ interchangeably.

***Note.*** The page limit prevents us to include the proofs, which are available in technical report [20].

## 2. PRELIMINARIES

The idea of solving problem (1) has the root in *dissipative power minimization* that naturally happens in the course of electric conduction over a media, and is mathematically explained by Dirichlet's Principle. As the first step to adopt this idea for a multiclass wireless network, in Sec. 3 we fantasize a so-called multiclass nonlinear resistive network, and generalize the concept of Dirichlet's

Principle and dissipative power minimization on it. Inspired by this, we propose in Sec. 4 a multiclass routing protocol that solves problem (1), proved by showing that under this protocol, the long-term average flow of packets on the wireless network takes the form of electric currents on its analogous nonlinear resistive network.

## 2.1 Stability and Throughput Optimality

A discrete-time stochastic process $x(n)$ is stable if

$$\overline{x} := \limsup_{\tau\to\infty} 1/\tau \sum\nolimits_{n=0}^{\tau-1} \mathbb{E}\{x(n)\} < \infty. \qquad (2)$$

The definition of stability and the overbar notation are extended entrywise to vectors and matrices. A network is *stable* if all its queues are stable. An arrival rate matrix is *stabilizable* if there exists a routing policy to stabilize the network. For a routing policy, *stability region* is the set of all arrival rate matrices that it can stably support. Network *capacity region* is the union of the stability regions of all possible routing policies (probably unfeasible). A routing policy is *throughput-optimal* if its stability region coincides with the network capacity region; thus it secures network stability for all stabilizable arrival rates.

## 2.2 Channel Interference

Contrary to wireline networks where links are independent resources, two wireless links cannot simultaneously transmit if they have interference. An interference model specifies these restrictions on simultaneous transmissions. Given an interference model, we define a *maximal schedule* as a set of channels such that no two channels interfere with each other, and no more channel can be added to it without violating the constraints of interference model. We describe a maximal schedule with a *scheduling vector* $\boldsymbol{\pi} \in \{0,1\}^{|\mathcal{E}|}$ where $\pi_{ij}$ takes the value 1 if the channel $ij$ is included in the maximal schedule, and 0 otherwise. Given a connectivity graph $(\mathcal{V},\mathcal{E})$, we also define the *scheduling set* $\Pi$ as the collection of all *maximal* scheduling vectors.

The scheduling set varies according to interference model. The results of this paper remain valid for the family of all interference models under which a node is not allowed to transmit to more than one neighbor at the same time. Thus, in a most general case, a node may receive packets from several neighbors while sending packets over one of its outgoing links. Interference constraints used with all well-known network and link layer protocols, including general K-hop interference models, fall in this family.

## 2.3 Time-Varying Topology

Network topology may vary in time due to node mobility and/or surrounding conditions, e.g. obstacle effect or channel fading. We assume that the sets $\mathcal{V}$ and $\mathcal{E}$ change far slower than channel states; thus we take them fixed during the time of interest. Then, an unavailable channel is characterized by a zero link capacity. Persistent variations, due to e.g. non-local mobility, can be caught in a long scale regime that updates connectivity graph $(\mathcal{V},\mathcal{E})$. We also assume that channel states remain fixed during a timeslot, while they may change across slots according to some (unknown) probability laws.

Let a stochastic process $\boldsymbol{S}(n) = \big(S_1(n),\cdots,S_{|\mathcal{E}|}(n)\big)$ represent channel states at slot $n$, describing all uncontrollable conditions that affect channel capacities, and possibly link cost-factors. Assume that $\boldsymbol{S}(n)$ evolves according to an ergodic stationary process and takes values in a finite (but arbitrarily large) set $\mathcal{S}$. For example, an irreducible Markov chain or any i.i.d. sequence of stochastic matrices are both ergodic and stationary. By Birkhoff's ergodic theorem, each state $\boldsymbol{S}\in\mathcal{S}$ is of probability

$$s := \mathbb{P}\big\{\boldsymbol{S}(n)=\boldsymbol{S}\big\} = \limsup_{\tau\to\infty} 1/\tau \sum\nolimits_{n=0}^{\tau-1} \mathbb{I}_{\boldsymbol{S}(n)=\boldsymbol{S}}$$

where $\sum_{\boldsymbol{S}\in\mathcal{S}} s = 1$. Though our proposed routing protocol does not require the state probabilities $s$, the existence of $s$ is important to establish the network capacity region, and also for the theoretical analysis of our protocol.

## 2.4 State Model of Multiclass Network

Let $q_i^{(d)}(n)$ represent the integer number of $d$-classes in the node $i$ at slot $n$. It is assumed that a packet leaves the network as soon as reaching its destination; thus the backlog of $d$-classes at the destination node $d$ is zero for all $d\in\mathcal{K}$. Then the state variables of the queuing system are represented by the hyper-vector

$$\boldsymbol{q}_\circ(n) := \big[\, \boldsymbol{q}_\circ^{(1)}(n),\ldots,\boldsymbol{q}_\circ^{(|\mathcal{K}|)}(n)\,\big]^\top \in \mathbb{R}^{(|\mathcal{V}|-1)|\mathcal{K}|}$$
$$\boldsymbol{q}_\circ^{(d)}(n) := \big[\, q_1^{(d)}(n),\ldots,q_{d-1}^{(d)}(n), q_{d+1}^{(d)}(n),\ldots,q_{|\mathcal{V}|}^{(d)}(n)\,\big]$$

where $q_d^{(d)}(n)\equiv 0$ is dropped from the set of states.

**Notation.** A subscript $\circ$ denotes a reduced array by discarding the entries related to the destination node $d$.

Let a stochastic process $a_i^{(d)}(n)$ be the number of exogenous $d$-classes arriving into the node $i$ at slot $n$. Discarding $a_d^{(d)}(n)\equiv 0$, the hyper-vector of node arrivals is

$$\boldsymbol{a}_\circ(n) := \big[\, \boldsymbol{a}_\circ^{(1)}(n),\ldots,\boldsymbol{a}_\circ^{(|\mathcal{K}|)}(n)\,\big]^\top \in \mathbb{R}^{(|\mathcal{V}|-1)|\mathcal{K}|}$$
$$\boldsymbol{a}_\circ^{(d)}(n) := \big[\, a_1^{(d)}(n),\ldots,a_{d-1}^{(d)}(n), a_{d+1}^{(d)}(n),\ldots,a_{|\mathcal{V}|}^{(d)}(n)\,\big].$$

For a link $ij\in\mathcal{E}$, the *capacity* $\mu_{ij}(n)$, which is frequently called link *transmission rate* in literature, counts the maximum number of packets the link can transmit at slot $n$. The link *actual-transmission* $f_{ij}^{(d)}(n)$, on the other hand, counts the number of $d$-packets *genuinely* sent over the link at slot $n$ — under a routing protocol. We form the hyper-vector of link actual-transmissions as

$$\boldsymbol{f\!\!f}(n) := \big[\, \boldsymbol{f}^{(1)}(n),\ldots,\boldsymbol{f}^{(|\mathcal{K}|)}(n)\,\big]^\top \in \mathbb{R}^{|\mathcal{E}||\mathcal{K}|}$$
$$\boldsymbol{f}^{(d)}(n) := \big[\, f_1^{(d)}(n),\ldots,f_{|\mathcal{E}|}^{(d)}(n)\,\big].$$

Given a directed graph $(\mathcal{V},\mathcal{E})$, let $\boldsymbol{B}$ denote the *node-edge* incidence matrix in which $B_{i\ell}$ — the entry related to node $i$ and edge $j$ — takes the value 1 if node $i$ is the tail of directed edge $\ell$, $-1$ if $i$ is the head, and 0 otherwise. For a class $d$, let $\boldsymbol{B}_\circ^{(d)}$ denote a reduction of $\boldsymbol{B}$ that discards the row related to the destination node $d$. Extending this structure to a multiclass framework, we get

$$\mathbb{B}_\circ := \operatorname{diag}\big(\,[\,\boldsymbol{B}_\circ^{(1)},\ldots,\boldsymbol{B}_\circ^{(|\mathcal{K}|)}\,]\,\big) \in \mathbb{R}^{(|\mathcal{V}|-1)|\mathcal{K}|\times|\mathcal{E}||\mathcal{K}|}.$$

One can then verify that $\mathbb{B}_\circ \boldsymbol{f}(n)$ is a hyper-vector in which the entry corresponding to node $i$ and class $d$ is

$$(\mathbb{B}_\circ \boldsymbol{f})_i^{(d)}(n) = \sum\nolimits_{b\in\text{out}(i)} f_{ib}^{(d)}(n) - \sum\nolimits_{a\in\text{in}(i)} f_{ai}^{(d)}(n)$$

where $\text{in}(i)$ and $\text{out}(i)$ respectively denote the set of incoming and outgoing neighbors of node $i$.

Using these ingredients, the $\boldsymbol{f}$-controlled, state dynamics of a multiclass queuing network is captured by

$$\boldsymbol{q}_\circ(n+1) = \boldsymbol{q}_\circ(n) + \boldsymbol{a}_\circ(n) - \mathbb{B}_\circ \boldsymbol{f}(n). \tag{3}$$

### 2.5 V-Parameter Back-Pressure (BP) Algorithm

In the original BP [9], at every timeslot, each link receives a weight as the product of its queue differential and its capacity, and then a set of non-interfering links with maximum cumulative weight are scheduled for the forwarding. To incorporate a cost function into the algorithm, the V-parameter BP [8] penalizes each link with its related cost via a user-assigned parameter $V\in[0,\infty)$ that determines the worthiness of reducing the cost function at the expense of increasing the network delay, while $V=0$ recovers the original BP.

To reduce the Dirichlet routing cost, defined in (1), as the cost function, at every timeslot $n$, the V-parameter BP observes queue backlogs $q_i^{(d)}(n)$, and estimates channel capacities $\mu_{ij}(n)$ and link cost factors $\rho_{ij}^{(d)}(n)$, to make a network-layer packet transmission decision as follows.

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**Weighing:** On each directed link $ij$ and for each class $d$ find $q_{ij}^{(d)}(n) := q_i^{(d)}(n) - q_j^{(d)}(n)$ and select the optimal class

$$d_{ij}^*(n) := \arg\max\nolimits_{d\in\mathcal{K}}\; q_{ij}^{(d)}(n). \tag{4}$$

Then give a weight to the link as

$$w_{ij}(n) := \mu_{ij}(n)\big(\,q_{ij}^{(d^*)}(n) - V\rho_{ij}^{(d)}(n)\,\mu_{ij}(n)\,\big)^+. \tag{5}$$

**Scheduling:** Find the scheduling vector such that

$$\boldsymbol{\pi}(n) = \arg\max_{\boldsymbol{\pi}\in\Pi} \sum\nolimits_{ij\in\mathcal{E}} \pi_{ij} w_{ij}(n) \tag{6}$$

where ties are broken randomly.

**Forwarding:** On each activated link $ij$ with $w_{ij}(n)>0$ transmit from the class $d_{ij}^*(n)$ at full capacity $\mu_{ij}(n)$.

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## 3. DIRICHLET'S PRINCIPLE

Consider a conducting medium $\mathcal{M}$ as a bounded continuous domain in a Euclidean space, with $A(\boldsymbol{z})$ being the current *source* injected into (with minus for the current drawn from) the point $\boldsymbol{z}\in\mathcal{M}$. Let $Q(\boldsymbol{z})$ be the induced voltage potential on $\mathcal{M}$, while being prescribed on the boundary $\partial\mathcal{M}$. Let vector function $\boldsymbol{F}(\boldsymbol{z})$ be the electrical current passing through the point $\boldsymbol{z}$. In the steady-state equilibrated conduction, the principle of charge conservation asserts that the current entering into any bounded region $\mathcal{M}'\subset\mathcal{M}$ must be equal to the current leaving out the region, i.e. $\int_{\partial\mathcal{M}'} \boldsymbol{F}(\boldsymbol{z}) \llcorner \aleph(\boldsymbol{z}) = \int_{\mathcal{M}'} A(\boldsymbol{z})$ where $\aleph(\boldsymbol{z})$ is the exterior normal, and $\llcorner$ the projection operator. By the Divergence theorem, $\int_{\mathcal{M}'} \operatorname{div}\boldsymbol{F}(\boldsymbol{z}) = \int_{\mathcal{M}'} A(\boldsymbol{z})$. As $\mathcal{M}'$ can be chosen infinitesimally small, the latter implies

$$\operatorname{div}\boldsymbol{F}(\boldsymbol{z}) = A(\boldsymbol{z}). \tag{7}$$

By Ohm's law, on the other hand, current between two points is proportional to the gradient of voltage across the points scaled by the *conductivity* of the material,

$$\boldsymbol{F}(\boldsymbol{z}) = -\sigma(\boldsymbol{z})\,\nabla Q(\boldsymbol{z}). \tag{8}$$

where the conductivity $\sigma(\boldsymbol{z})$ is in general a positive definite symmetric matrix. Substituting (8) into (7), we get

$$\operatorname{div}\big(\sigma(\boldsymbol{z})\,\nabla Q(\boldsymbol{z})\big) + A(\boldsymbol{z}) = 0 \tag{9}$$

which formulates the classical Poisson equation.

Dirichlet's Principle states that Poisson's equation (9) has a unique solution that minimizes the Dirichlet energy

$$E\big(Q(\boldsymbol{z})\big) := \int_{\mathcal{M}} \Big(\,\frac{1}{2}\,\sigma\|\nabla Q(\boldsymbol{z})\|^2 - Q(\boldsymbol{z})A(\boldsymbol{z})\Big)$$

among all twice differentiable functions $Q(\boldsymbol{z})$ that respect the prescribed voltage potential on the boundary $\partial\mathcal{M}$ [1].

### 3.1 Prelude: Linear Resistive Networks

Rather than a smoothly distributed conductor $\mathcal{M}$, consider now a resistive network $(\mathcal{V},\mathcal{E})$ in which two neighboring nodes are connected via a linear lumped resistor. Exogenous current is injected into (resp., drawn from) the network via positive (resp., negative) current sources attached to different nodes. Assume that voltage is fixed to ground at a single node as *reference,* also referred to as *sink,* which is analogous to a collapse of the boundary $\partial\mathcal{M}$ to a point on the continuous domain. One may visualize the reference as the node that collects the net current injected into the network, i.e. algebraic sum of current sources, and drains it back into the sources, so that building a *closed* system.

To solve circuit problems, it is essential to assign an *arbitrary* orientation to each edge with the understanding that the particular choice of orientation has no impact on the solutions. Accordingly, every edge variable is signed, while a negative quantity is interpreted to be on the opposite direction of the edge orientation. Using the notion of arbitrary orientation, the node-edge incident matrix $\boldsymbol{B}$, previously defined on a directed graph, is also defined on the undirected graph here in the same way.

Let $\boldsymbol{q}$ represent the vector of node voltages, $\boldsymbol{f}$ the vector of edge currents, $\boldsymbol{\sigma}$ the vector of edge conductances, and $\boldsymbol{a}$ the vector of node current sources. With $d$ being the reference node, we assume $a_d\equiv 0$ and define the reduced arrays $\boldsymbol{q}_\circ$, $\boldsymbol{a}_\circ$ and $\boldsymbol{B}_\circ$ through discarding the entries related to the node $d$. Then the principle of charge conservation in (7) becomes Kirchhoff's Current Law (KCL) on the network, asserting that at each non-grounded node, the algebraic sum of currents must be zero,

$$\boldsymbol{B}_\circ \boldsymbol{f} = \boldsymbol{a}_\circ. \tag{10}$$

Ohm's law in (8), on the other hand, becomes

$$\boldsymbol{f} = \mathrm{diag}(\boldsymbol{\sigma})\, \boldsymbol{B}_\circ^\top \boldsymbol{q}_\circ. \tag{11}$$

Substituting (11) into (10), we get

$$-\boldsymbol{L}_\circ \boldsymbol{q}_\circ + \boldsymbol{a}_\circ = \boldsymbol{0} \quad \text{with} \quad \boldsymbol{L}_\circ := \boldsymbol{B}_\circ \mathrm{diag}(\boldsymbol{\sigma}) \boldsymbol{B}_\circ^\top \tag{12}$$

as the graph combinatorial analog of the classical Poisson equation. The matrix $\boldsymbol{L}_\circ$, called the *Dirichlet Laplacian*, is *symmetric positive definite* for a connected network.

Like the classical case, (12) has a unique solution that minimizes the combinatorial Dirichlet energy

$$E(\boldsymbol{q}_\circ) := \frac{1}{2}\, \boldsymbol{q}_\circ^\top \boldsymbol{L}_\circ \boldsymbol{q}_\circ - \boldsymbol{q}_\circ^\top \boldsymbol{a}_\circ. \tag{13}$$

The proof is much simpler in the combinatorial case and is directly concluded from the positive definiteness of $\boldsymbol{L}_\circ$.

An important implication of Dirichlet's Principle is the minimization of *power dissipation* on a linear resistive network. For a vector of currents $\boldsymbol{f}$ subject to KCL (10), dissipative energy $E_R(\boldsymbol{f}) := \boldsymbol{f}^\top \mathrm{diag}(\boldsymbol{\sigma})^{-1} \boldsymbol{f}$, let $\boldsymbol{f}^\star$ be the configuration of currents that minimize $E_R(\boldsymbol{f})$, and $\boldsymbol{q}^\star$ the configuration of voltages that minimize $E(\boldsymbol{q}_\circ)$ in (13). Then it is not difficult to shown that the $\boldsymbol{f}^\star$ and $\boldsymbol{q}^\star$ are related to each other by Ohm's law (11).

## 3.2 Capacitated Directed Networks

Rather than an undirected network with free-capacity edges, consider now a network under both edge directionality and edge capacity constraints. The electrical network is of the same configuration as that in the linear resistive network with the exception that here, rather than using a linear resistor, we connect two neighboring nodes using a nonlinear resistor in series with an ideal diode. The nonlinear resistor limits the current to the edge capacity, while the ideal diode allows the current only along the edge direction. For an edge $ij$ with capacity $\mu_{ij}$, the characteristic of nonlinear resistor is given by

$$r_{ij} = \begin{cases} 1/\sigma_{ij} & \text{if } |q_{ij}| \leqslant \mu_{ij}/\sigma_{ij} \\ |q_{ij}|/\mu_{ij} & \text{if } |q_{ij}| > \mu_{ij}/\sigma_{ij} \end{cases}$$

where $\sigma_{ij}$ is the conductance in linear regime when the current is below the edge capacity (see Fig. 1).

On any electrical network, the KCL in (10) remains unchanged. However, on our nonlinear resistive network, Ohm's law (11), however, must be modified to allow the current in only one direction, and to limit it within the edge capacity. Let the arbitrarily-chosen edge orientations concur with the edge directions. Then, with $\boldsymbol{\mu}$ being the vector of edge capacities, the modified Ohm law becomes

$$\boldsymbol{f} = \min\big\{\mathrm{diag}(\boldsymbol{\sigma})\big(\boldsymbol{B}_\circ^\top \boldsymbol{q}_\circ\big)^+,\ \boldsymbol{\mu}\big\}. \tag{14}$$

Plugging (14) in (10) leads to the analogous of Poisson's equation on a capacity-constrained directed network as

$$\begin{gathered} -\vec{\boldsymbol{L}}_\circ(\boldsymbol{q}_\circ) + \boldsymbol{a}_\circ = \boldsymbol{0} \\ \vec{\boldsymbol{L}}_\circ(\boldsymbol{q}_\circ) := \boldsymbol{B}_\circ \min\big\{\mathrm{diag}(\boldsymbol{\sigma})\big(\boldsymbol{B}_\circ^\top \boldsymbol{q}_\circ\big)^+,\ \boldsymbol{\mu}\big\}. \end{gathered} \tag{15}$$

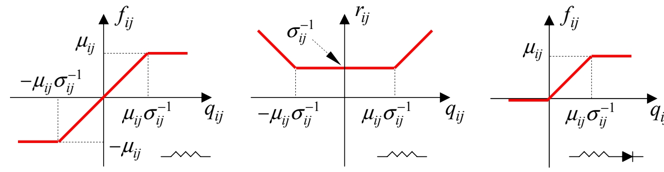


**Figure 1: The nonlinear resistive edge with an ideal diode: (left) The current-voltage curve of the resistor. (middle) The resistive-voltage curve of the resistor. (right) The current-voltage curve of the resistor and diode together.**

We call $\vec{\boldsymbol{L}}_\circ(\cdot)$ as *nonlinear Dirichlet Laplacian* operator.

Contrary to the standard Laplacian $\boldsymbol{L}_\circ$ on a linear resistive network, $\vec{\boldsymbol{L}}_\circ$ here is an operand-dependent operator that retains neither linearity nor symmetry. Thus the easy way of proving Dirichlet's Principle on free-capacity undirected networks ceases to exist here, as we can no longer claim that $\vec{\boldsymbol{L}}_\circ \boldsymbol{q}_\circ$ in (15) is the directional derivative of $\frac{1}{2}\, \boldsymbol{q}_\circ^\top \vec{\boldsymbol{L}}_\circ \boldsymbol{q}_\circ$ along $\boldsymbol{q}_\circ$. Nonetheless, the next theorem extends the concept of Dirichlet's Principle, and from there the merit of minimizing dissipative energy, to capacity-constrained directed networks.

**Theorem 1.** *Consider a capacitated directed network under a feasible vector of current sources $\boldsymbol{a}_\circ$, i.e. there exists at least one configuration of currents $\boldsymbol{f}$ that satisfy KCL at nodes. Then the nonlinear Poisson equation* (15) *has a unique solution that minimizes the Dirichlet-like energy*

$$\vec{E}(\boldsymbol{q}_\circ) := \frac{1}{2}\, \boldsymbol{q}_\circ^\top\, \vec{\boldsymbol{L}}_\circ(\boldsymbol{q}_\circ) - \boldsymbol{q}_\circ^\top \boldsymbol{a}_\circ. \tag{16}$$

*Further, minimizing $\vec{E}(\boldsymbol{q}_\circ)$ is equivalent to solving*

$$\begin{aligned} &\textit{Minimize:} && \vec{E}_R(\boldsymbol{f}) := \boldsymbol{f}^\top \mathrm{diag}(\boldsymbol{\sigma})^{-1} \boldsymbol{f} \\ &\textit{Subject to:} && \textit{1)}\ \ \boldsymbol{0} \preccurlyeq \boldsymbol{f} \preccurlyeq \boldsymbol{\mu} \\ & && \textit{2)}\ \ \boldsymbol{B}_\circ \boldsymbol{f} = \boldsymbol{a}_\circ \end{aligned} \tag{17}$$

*which formulates the minimization of dissipative energy subject to the network constraints and KCL at nodes.*

## 3.3 Multiclass Networks

In a traditional electrical network, the total net charge generated by all current sources is absorbed by one single grounded node — the sink. A more complex scenario, however, may be fantasized in parallel with multiclass problems in data networking. Specifically, consider a setting in which different types of charges are generated by current sources, where each type of charge is absorbed by a specific node as the sink of that charge. Using a similar terminology, let us refer to each type of charge as a *class*.

In the absence of edge capacity constraints, each class has its own independent conduction, so that the multiclass network can be viewed as the collection of fully decoupled uniclass networks with no mutual impact. When the edges are of limited capacities, however, the conduction of different classes no longer happens independently, because the way of allocating edge capacities to each class has a direct impact on the conduction of that class, while the sum of allocated capacities on each edge is bounded. For example, allocating the total capacity of one edge to only one class means eliminating that edge for all other classes.

Consider now a multiclass electrical network $(\mathcal{V}, \mathcal{E}, \mathcal{K})$ subject to both edge directionality and edge capacity constraints. Let $0 \leqslant \theta_{ij}^{(d)} \leqslant 1$ represent the portion of total capacity of the edge $ij$ devoted to the class $d$, i.e.,

$$\mu_{ij}^{(d)} = \theta_{ij}^{(d)} \mu_{ij} \ \text{ with } \ \textstyle\sum_{d\in\mathcal{K}} \theta_{ij}^{(d)} \leqslant 1. \tag{18}$$

We accordingly form $\boldsymbol{\theta}^{(d)} \in \mathbb{R}^{|\mathcal{E}|}$ as the vector of edge capacity factors for the class $d$. Endowing edges with the possibility of having different conductivities for different classes, we also let $\boldsymbol{\sigma}^{(d)} \in \mathbb{R}^{|\mathcal{E}|}$ be the vector of $d$-conductivity on edges. If one can figure out the vector $\boldsymbol{\theta}^{(d)}$ for each class $d$, then the conduction of each class will readily comply with the uniclass equations (14)–(15). To have a compact formulation, let us form the *hyper-vectors* $\boldsymbol{q}_\circ$, $\boldsymbol{a}_\circ$, and $\boldsymbol{f\!\!f}$ conformably structured as their counterparts defined in Sec. 2.4, and also $\boldsymbol{\sigma}$ and $\boldsymbol{\theta}$ in a similar way. Then the steady-state electric conduction on a multiclass capacity-constrained directed network is described by

$$\mathbb{B}_\circ \boldsymbol{f\!\!f} = \boldsymbol{a}_\circ. \tag{19}$$

$$\boldsymbol{f\!\!f} = \min\big\{\operatorname{diag}(\boldsymbol{\sigma})\big(\mathbb{B}_\circ^\top \boldsymbol{q}_\circ\big)^+, \boldsymbol{\theta}\odot(\mathbf{1}_{|\mathcal{K}|}\otimes\boldsymbol{\mu})\big\} \tag{20}$$

$$\begin{gathered} -\vec{\mathbb{L}}_\circ(\boldsymbol{q}_\circ) + \boldsymbol{a}_\circ = \mathbf{0} \\ \vec{\mathbb{L}}_\circ(\boldsymbol{q}_\circ) := \mathbb{B}_\circ \min\big\{\operatorname{diag}(\boldsymbol{\sigma})\big(\mathbb{B}_\circ^\top \boldsymbol{q}_\circ\big)^+, \boldsymbol{\theta}\odot(\mathbf{1}_{|\mathcal{K}|}\otimes\boldsymbol{\mu})\big\}. \end{gathered} \tag{21}$$

The term $(\mathbf{1}_{|\mathcal{K}|}\otimes\boldsymbol{\mu})$ extends $\boldsymbol{\mu} \in \mathbb{R}^{|\mathcal{E}|}$ to be of size $|\mathcal{E}||\mathcal{K}|$ and so can be used in a multiclass fashion, where its entry-wise product with $\boldsymbol{\theta}$ shapes (18) in a hyper-vector form.

To answer the crucial question of how to allocate edge capacities to different classes, we first introduce a key property of the uniclass Ohm law in the next theorem.

**Theorem 2.** *On a uniclass resistive network with capacity-free undirected edges (resp., capacity-constrained directed edges), the electrical current assigned by the linear Ohm law* (11) *(resp., by the nonlinear Ohm law* (14)*) uniquely minimizes the functional* $\|\operatorname{diag}(\boldsymbol{\sigma})\boldsymbol{B}_\circ^\top \boldsymbol{q}_\circ - \boldsymbol{f}\|$ *among all admissible currents that respect KCL at nodes.*

Extending this result to a multiclass resistive network, the vector of multiclass currents $\boldsymbol{f\!\!f}$ must minimize the multiclass functional $\|\operatorname{diag}(\boldsymbol{\sigma})\mathbb{B}_\circ^\top \boldsymbol{q}_\circ - \boldsymbol{f\!\!f}\|$. In the absence of edge capacity constraints, this is readily concluded from Th. 2 together with the flow independency among different classes. Under limited edge capacities, however, the configuration of $\boldsymbol{f\!\!f}$ depends on the configuration of edge capacity factors $\boldsymbol{\theta}$; thus the minimizing $\boldsymbol{f\!\!f}$ determines $\boldsymbol{\theta}$,

$$\begin{gathered} \boldsymbol{\theta} = \arg\min_{\boldsymbol{\theta}} \|\operatorname{diag}(\boldsymbol{\sigma})\mathbb{B}_\circ^\top \boldsymbol{q}_\circ - \boldsymbol{f\!\!f}\| \\ \text{subject to: } \textstyle\sum_{d\in\mathcal{K}} \boldsymbol{\theta}^{(d)} \preccurlyeq \mathbf{1}_{|\mathcal{E}|}. \end{gathered} \tag{22}$$

While the optimal $\boldsymbol{f\!\!f}$ that solves (22) is unique, the related $\boldsymbol{\theta}$ is not necessarily unique, i.e. different $\boldsymbol{\theta}$ may lead to the same optimal $\boldsymbol{f\!\!f}$. In Fig 2, for example, it is easy to confirm that to solve (22), two units of current destined for node $a$ (resp., node $b$) should be sent via edge $sa$ (resp., edge $sb$) and one unit via edges $sc$ and $ca$ (resp., edges $sc$ and $cb$). Thus any division of edge capacities

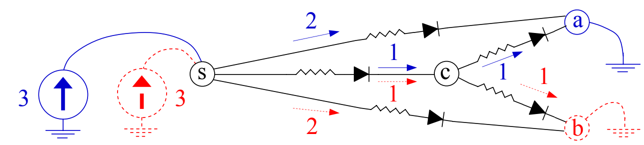


**Figure 2: Node $s$ injects two classes of electrical currents with intensity 3, one destined for node $a$ and the other for node $b$. For all resistors $\sigma_{ij} = 1$, and for all edges $\mu_{ij} = 5$.**

between the two classes is admissible as far as it provides for class $a$ (resp., class $b$) the capacity of at least two on edge $sa$ (resp., edge $sb$) and the capacity of at least one on edges $sc$ and $ca$ (resp., edges $sc$ and $cb$).

The upshot of this section is the next theorem that extends the concept of Dirichlet's Principle, and from there the merit of minimizing dissipative energy, to multiclass conduction on capacity-constrained directed networks.

**Theorem 3.** *Consider a multiclass capacitated directed network under a feasible vector of current sources* $\boldsymbol{a}_\circ$*, i.e. there exists at least one configuration of multiclass currents* $\boldsymbol{f\!\!f}$ *that satisfy the multiclass KCL at nodes. Then the nonlinear Poisson equation* (21) *has a unique solution that minimizes the multiclass Dirichlet-like energy*

$$\vec{E}(\boldsymbol{q}_\circ) := \frac{1}{2}\, \boldsymbol{q}_\circ^\top \, \vec{\mathbb{L}}_\circ(\boldsymbol{q}_\circ) - \boldsymbol{q}_\circ^\top \boldsymbol{a}_\circ. \tag{23}$$

*Further, minimizing* $\vec{E}(\boldsymbol{q}_\circ)$ *under the edge capacity allocation* (22) *is equivalent to solving*

$$\begin{aligned} &\textit{Minimize:} && \vec{E}_R(\boldsymbol{f\!\!f}) := \boldsymbol{f\!\!f}^\top \operatorname{diag}(\boldsymbol{\sigma})^{-1} \boldsymbol{f\!\!f} \\ &\textit{Subject to:} && \textit{1)}\ \mathbf{0} \preccurlyeq \textstyle\sum_{d\in\mathcal{K}} \boldsymbol{f}^{(d)} \preccurlyeq \boldsymbol{\mu} \\ & && \textit{2)}\ \mathbb{B}_\circ \boldsymbol{f\!\!f} = \boldsymbol{a}_\circ \end{aligned} \tag{24}$$

*which formulates the minimization of dissipative energy subject to the network constraints and the multiclass KCL.*

## 4. DIRICHLET-BASED ROUTING PROTOCOL

On a multiclass data network, BP-based schemes transmit packets from only one class over each activated link at each timeslot. First, when the number of packets from individual classes is not enough to fill up the link capacities, network resources are squandered by this single class transmission policy. In other words, the larger capacity of network would be utilized, and so the average network delay would decrease, if the capacity of each activated link were properly filled up with packets from different classes. Second, even if the links could be stuffed with individual classes, still, as shown in [19], blindly sending the maximum number of packets from only one class on each link would merely deplete the network resources with even negative impact on delay performance.

Our proposed multiclass routing protocol here is an answer to the question of how a dynamic routing policy, with no routing path constraint, can effectively utilize the maximum timeslot resources. Our solution is inspired by the multiclass electric conduction developed in the previous section, telling us that a dissipative energy minimizing policy is supposed to send different classes on each activated link. Further, the optimal capacity allocation (22) suggests that a class $d$ should receive a piece of capacity of edge $ij$ proportional to $\sigma_{ij}^{(d)} q_{ij}^{(d)}$, i.e. its queue differential scaled by its related link profit-factor — reciprocal of the link cost-factor.

The new routing protocol has the same algorithmic structure, complexity, and overhead as BP policy, which provides a convenient way of unifying it with the previous works based on BP. At every timeslot $n$, it observes queue backlogs $q_i(n)$, and estimates channel capacities $\mu_{ij}(n)$ and link cost factors $\rho_{ij}^{(d)}(n)$, to make a network-layer packet transmission decision as follows.

---

**Weighing:** On every directed link $ij$ and for each class $d$ find $q_{ij}^{(d)}(n) := q_i^{(d)}(n) - q_j^{(d)}(n)$ and create a set

$$\mathcal{K}_{ij}(n) \subseteq \mathcal{K} \text{ such that } q_{ij}^{(d)}(n) > 0\,,\ \forall d \in \mathcal{K}_{ij}(n).$$

Fix $\widehat{f_{ij}^{(d)}}(n) = 0$ for each $d \notin \mathcal{K}_{ij}(n)$, and first find $\widehat{f_{ij}^{(d)}}(n)$ for every $d \in \mathcal{K}_{ij}(n)$ by solving the optimization problem

$$\begin{aligned} &\text{Minimize: } \textstyle\sum_{d\in\mathcal{K}_{ij}(n)} \Big(\rho_{ij}^{(d)}(n)^{-1}\, q_{ij}^{(d)}(n) - \widehat{f_{ij}^{(d)}}(n)\Big)^2 \\ &\text{Subject to: } \begin{cases} \sum_{d\in\mathcal{K}_{ij}(n)} \widehat{f_{ij}^{(d)}}(n) \leqslant \mu_{ij}(n) \\ 0 \leqslant \widehat{f_{ij}^{(d)}}(n) \leqslant q_{ij}^{(d)}(n),\ \forall d \in \mathcal{K}_{ij}(n) \end{cases} \end{aligned} \tag{25}$$

where $\widehat{f_{ij}^{(d)}}(n)$ denotes the number of packets the link would transmit if it were activated — thus a predicted value which would not necessarily be realized. Then give a weight to each class $d \in \mathcal{K}_{ij}(n)$ as

$$w_{ij}^{(d)}(n) := 2\,\rho_{ij}^{(d)}(n)^{-1}\, q_{ij}^{(d)}(n)\, \widehat{f_{ij}^{(d)}}(n) - \Big(\widehat{f_{ij}^{(d)}}(n)\Big)^2 \tag{26}$$

and aggregate them to determine the final link weight as

$$w_{ij}(n) := \textstyle\sum_{d\in\mathcal{K}_{ij}(n)} w_{ij}^{(d)}(n). \tag{27}$$

**Scheduling:** Find the scheduling vector, in the same way as BP, using the max-weight scheduling (6).
**Forwarding:** On each activated link $ij$, transmit $\widehat{f_{ij}^{(d)}}(n)$ number of packets from the class $d$.

---

Discriminating link transmission predictions $\widehat{f_{ij}^{(d)}}(n)$, link actual transmissions $f_{ij}^{(d)}(n)$, and link capacities $\mu_{ij}(n)$ from each other is crucial to understand the algorithm. Another point is that like BP, our algorithm also rests on a centralized scheduling whose complexity can be prohibitive in practice. Fortunately, much progress has been made to ease this difficulty by designing decentralized schedulers with an arbitrary tradeoff between complexity and closeness to the centralized performance [21].

Problem (25) is a standard least-norm optimization with variable bounds that can be solved in fast polynomial time at each node, i.e. in a fully decentralized manner. A related algorithm is developed as follows.

To simplify the notation, let us drop the overhat symbol and the time variable $(n)$. First observe that

$$\text{if } \textstyle\sum_{d\in\mathcal{K}_{ij}} q_{ij}^{(d)}/\rho_{ij}^{(d)} \leqslant \mu_{ij} \text{ then } f_{ij}^{(d)} = q_{ij}^{(d)}/\rho_{ij}^{(d)}$$

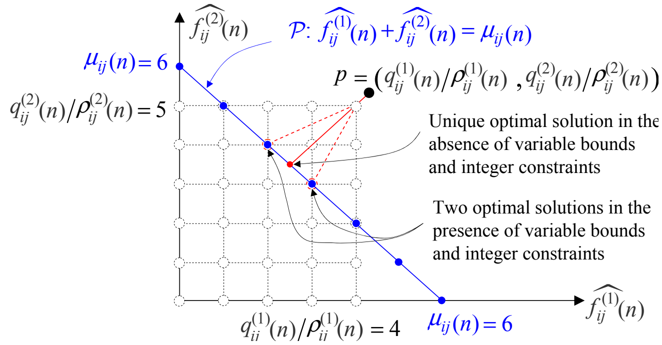


**Figure 3: Geometry of solving** (25) **for a two-class case when** $q_{ij}^{(1)}(n)/\rho_{ij}^{(1)}(n) + q_{ij}^{(2)}(n)/\rho_{ij}^{(2)}(n) > \mu_{ij}(n)$.

for each $d \in \mathcal{K}_{ij}$, and the problem is solved. Thus let

$$\textstyle\sum_{d\in\mathcal{K}_{ij}} q_{ij}^{(d)}/\rho_{ij}^{(d)} > \mu_{ij}\,.$$

This converts the first constraint from inequality into equality, viz. $\sum_{d\in\mathcal{K}_{ij}} f_{ij}^{(d)} = \mu_{ij}$. Then a basic Lagrange argument shows that in the absence of lower variable bounds and integer constraints, the problem has the following unique solution for each $d \in \mathcal{K}_{ij}$ :

$$f_{ij}^{(d)} = q_{ij}^{(d)}/\rho_{ij}^{(d)} + \big(\mu_{ij} - \textstyle\sum_{d\in\mathcal{K}_{ij}} q_{ij}^{(d)}/\rho_{ij}^{(d)}\big)/|\mathcal{K}_{ij}|.$$

By geometry, the latter reads the projection of the point $p := (\,q_{ij}^{(1)}/\rho_{ij}^{(1)}, \cdots, q_{ij}^{(|\mathcal{K}_{ij}|)}/\rho_{ij}^{(|\mathcal{K}_{ij}|)})$ onto the hyperplane

$$\mathcal{P}:\ \textstyle\sum_{d\in\mathcal{K}_{ij}} f_{ij}^{(d)} = \mu_{ij}\,.$$

Under integer constraints, $\mathcal{P}$ becomes an integer hypergrid and the optimal solution(s) will be the vertex(es) of this hypergrid with the shortest Euclidean distance to the point $p$. Note that the solution to the integer problem is not necessarily unique. Subjecting the solution to the lower variable bounds, it must also meet $f_{ij}^{(d)} \geqslant 0$, $\forall\, d \in \mathcal{K}_{ij}$. Based on this procedure, the following algorithm solves (25) for $\sum_{d\in\mathcal{K}_{ij}} q_{ij}^{(d)}/\rho_{ij}^{(d)} > \mu_{ij}$, while Fig. 3 gives a graphical demonstration for a two-class case.

---

S1: Let $h = \big(\sum_{d\in\mathcal{K}_{ij}(n)} \rho_{ij}^{(d)}(n)^{-1} q_{ij}^{(d)}(n) - \mu_{ij}(n)\big)/|\mathcal{K}_{ij}(n)|$ and $\forall d \in \mathcal{K}_{ij}(n)$ take $\widehat{f_{ij}^{(d)}}(n) = \rho_{ij}^{(d)}(n)^{-1} q_{ij}^{(d)}(n) - h$.

S2: Find $d_1 = \arg\min_{d\in\mathcal{K}_{ij}(n)} \widehat{f_{ij}^{(d)}}(n)$ and if $\widehat{f_{ij}^{(d_1)}}(n) < 0$, then remove $d_1$ from $\mathcal{K}_{ij}(n)$ and go back to S1.

S3: Let $r = \mu_{ij}(n) - \sum_{d\in\mathcal{K}_{ij}(n)} \lfloor \widehat{f_{ij}^{(d)}}(n) \rfloor$. For $r$ randomly chosen classes in $\mathcal{K}_{ij}(n)$ assign $\widehat{f_{ij}^{(d)}}(n) = \lceil \widehat{f_{ij}^{(d)}}(n) \rceil$ and for other classes in $\mathcal{K}_{ij}(n)$ assign $\widehat{f_{ij}^{(d)}}(n) = \lfloor \widehat{f_{ij}^{(d)}}(n) \rfloor$.

---

Observe that S1 finds the optimal solution in the absence of variable bounds and integer constraints, S2 ensures that the solution meets the variable lower bounds, and S3 determines an integer solution by finding a vertex on the integer hypergrid $\mathcal{P}$ with the shortest distance to the initial solution obtained by S1-S2. The term "$r$ randomly chosen classes" in S3 comes due to the fact that the integer problem may have more than one solution. When the initial solution of S1-S2 is integer, it will be the solution to the integer problem too, and so unique. Otherwise, there potentially exist several vertices on the integer hypergrid with equal distance from the non-integer initial solution and shorter than the distance of all other vertices. This is best shown in Fig. 3.

We remark that on a uniclass network, link transmission predictions and link weights are simplified to

$$\widehat{f_{ij}}(n) := \min\{\rho_{ij}(n)^{-1} q_{ij}(n)^+,\ \mu_{ij}(n)\}$$
$$w_{ij}(n) := 2\,\rho_{ij}(n)^{-1} q_{ij}(n)\,\widehat{f_{ij}}(n) - \big(\widehat{f_{ij}}(n)\big)^2.$$

This recovers our Pareto-optimal HD policy with $\beta = 1$, proposed for uniclass networks in [18].

## 5. ANALYSIS OF OPTIMAL PERFORMANCE

In Sec. 3, we developed multiclass nonlinear electric conduction on resistive networks with capacity-constrained directed edges. In Sec. 4, on the other hand, we developed a routing protocol for multiclass interfering wireless networks. The former describes a deterministic continuous-time process, while the latter leads to a stochastic time-slotted process. This section shows how these two seemingly different problems are rigorously correlated with each other. It specifically shows that in a long-term average basis, packet flow on a wireless network governed by our proposed routing protocol complies with electrical conduction on a suitably-defined resistive network, where the pivot is the notion of fluid limit. We also show the throughput optimality of our routing protocol and its delay minimization performance under uniform link cost-factors.

### 5.1 Throughput Optimality

Throughput optimality, as defined in Sec. 2.1, is an important quality factor of our proposed routing protocol.

**Theorem 4.** *Consider a stochastic multiclass wireless network with arrivals and channel states being i.i.d. random variables over timeslots and with respect to each other, and subject to an interference model that prohibits transmission to more than one neighbor at a timeslot. Then our proposed routing protocol in Sec. 4 is throughput-optimal in the sense that it secures network stability for any stabilizable matrix of arrival rates.*

### 5.2 Minimum Cost Routing

Fluid limit of a stochastic process is the limit dynamics obtained by *scaling* in time and amplitude. Under very mild conditions, it is shown that these scaled trajectories converge to a set of deterministic equations called *fluid model.* Using this deterministic model, one can analyze the *rate-level,* rather than *packet-level,* behavior of the original stochastic process. For the details, we refer the interested reader to [22] and references therein.

Let $\boldsymbol{X}(\omega, t)$ be a realization of a continuous-time stochastic process $\boldsymbol{X}$ along an arbitrary sample path $\omega$, and define the scaled process $\boldsymbol{X}^r(\omega, t) := \boldsymbol{X}(\omega, rt)/r$ for any $r > 0$. A deterministic function $\tilde{\boldsymbol{X}}(t)$ is called a *fluid limit* if there exist a sequence $r$ and a sample path $\omega$ such that $\lim_{r\to\infty} \boldsymbol{X}^r(\omega, t) \to \tilde{\boldsymbol{X}}(t)$ uniformly on compact sets. For a stable queuing network, the existence of fluid limits is guaranteed if exogenous arrivals are of finite variance. It is further shown that each fluid limit is Lipschitz-continuous, and so differentiable, almost everywhere with respect to Lebesgue measure on $[0, \infty)$ [22, 23]. Note that the fluid limit is independent of sample path.

**Theorem 5.** *Consider a stochastic multiclass wireless network under a stabilizable vector of packet arrivals $\boldsymbol{a}_\circ(n)$, and subject to an interference model that prohibits transmission to more than one neighbor at a timeslot. Suppose that the traffic is governed by our proposed routing protocol in Sec. 4. Then the fluid model of the network is described by the conduction equations* (19)–(22) *where in those equations, $\boldsymbol{a}_\circ$ is replaced by the expected time average of packet arrivals, $\boldsymbol{\mu}$ by the expected time average of link capacities, and $\boldsymbol{\sigma}$ by the inverse of the expected time average of link cost-factors from the wireless network, and where the expected time average is defined in* (2).

This theorem together with Th. 3 lead to the goal of this paper on the minimum cost routing.

**Corollary 1.** *Consider a stochastic multiclass wireless network under a stabilizable vector of packet arrivals $\boldsymbol{a}_\circ(n)$, and subject to an interference model that prohibits transmission to more than one neighbor at a timeslot. Then our proposed routing protocol in Sec. 4 solves the minimum cost routing problem* (1).

To our knowledge, this is the first time a feasible network-layer routing policy asserts the strict minimization of a routing penalty subject to network stability. Note that in the V-parameter BP [8], the $[O(V), O(1/V)]$ delay-cost tradeoff prevents minimizing the average routing cost subject to network stability, i.e. finite queue congestion.

### 5.3 Average Network Delay Minimization

**Theorem 6.** *Consider a stochastic multiclass wireless network with arrivals and channel states being i.i.d. random variables over timeslots and with respect to each other, and subject to an interference model that prohibits transmission to more than one neighbor at a timeslot. Suppose that all wireless links are of unit cost factor. Consider a class of network-layer routing policies that act based only on current queue congestion and current channel states. Within this class, our proposed routing protocol in Sec. 4 minimizes the average network delay by solving*

$$\begin{aligned} &\textit{Minimize:} \quad \limsup_{\tau\to\infty} \frac{1}{\tau} \sum_{n=0}^{\tau-1} \mathbb{E}\Big\{ \sum_{i\in\mathcal{V}} \sum_{d\in\mathcal{K}} q_i^{(d)}(n) \Big\} \\ &\textit{Subject to:} \quad 0 \leqslant \textstyle\sum_{d\in\mathcal{K}} f_{ij}^{(d)}(n) \leqslant \mu_{ij}(n),\ \forall ij \in \mathcal{E}. \end{aligned} \tag{28}$$

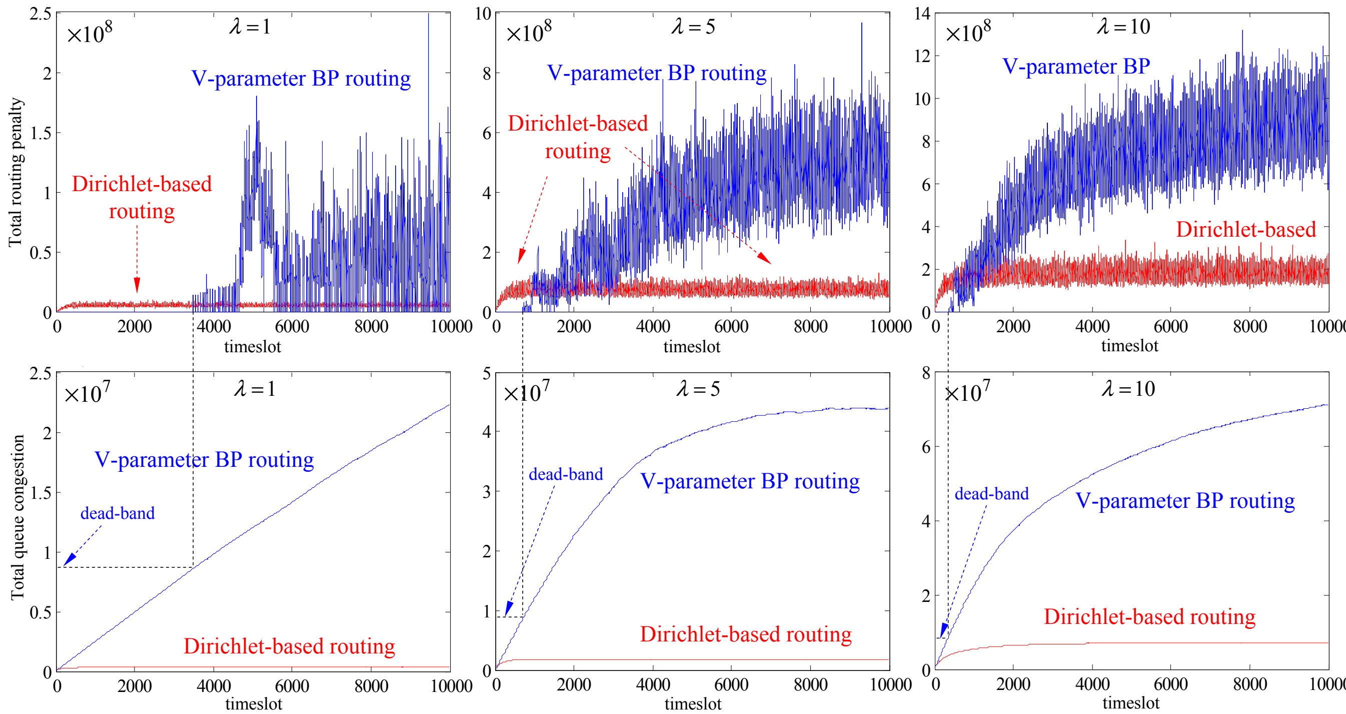


**Figure 4: Comparing the performance of our proposed routing protocol and V-parameter BP with V=0.8.**

Note that by Little's Theorem, for a given packet arrival rate, the expected time average total queue congestion in (28) is proportional to long-term average end-to-end network delay. Hence, minimizing (28) indeed ensures minimizing average network delay. In the light of Th. 5, this result should not be very surprising, as when all the links are of equal cost factors, the minimization of the Dirichlet routing cost is equivalent to the minimization of average total routing path on the network, which is closely related to the average network delay.

## 6. SIMULATION RESULTS

We consider a wireless network with 50 nodes randomly distributed on a surface. Links are placed between every two nodes whose proximity distance is less than a threshold, and extra links are added to make the network connected. Links are considered as two-way wireless channels, i.e. for any directed link $ij \in \mathcal{E}$ there exists $ji \in \mathcal{E}$ with the same capacity. The network runs under 1-hop interference model, i.e. links with common node cannot transmit at the same time. At every timeslot, the capacity of each link $ij$ follows a Gaussian distribution with the mean $m_{ij}$ and the variance equal to 150. To assign $m_{ij}$ to different links, we adopt Shannon capacity with power transmission $P_{ij}$, noise intensity $N_{ij}$, and a bandwidth of 1500, viz. $m_{ij} = 1500 \log_2(1 + P_{ij}/N_{ij})$. We randomly assign a noise intensity $N_{ij} \in [1, 5]$ to each link at first and keep it fixed during the simulation.

Assume that every node sends packets to every other node, forming a multiclass multihop wireless network. Different classes are generated at each node following Poisson's random variables with parameter $\lambda$, where all of them are i.i.d. over timeslots and with respect to each other. To support this traffic, we assume that each node can expend 30 units of transmission power per timeslot, which under 1-hop interference model leads to $P_{ij} = 30$. Each link receives a cost factor $\rho_{ij} \in [1, 10]$ for all classes at first that remains constant during the simulation.

The top panels in Fig. 4 display the timeslot evolution of total routing penalty $R(n) := \sum_{ij\in\mathcal{E}} \sum_{d\in\mathcal{K}} \rho_{ij} \big(f_{ij}^{(d)}(n)\big)^2$ for three arrival rates corresponding to the Poisson parameters $\lambda = 1$, 5, 10 packets per timeslot, comparing the performance of our routing protocol and V-parameter BP with $V = 0.8$. Note that the Dirichlet routing cost, defined in (1), is the expected time average of $R(n)$, where the expected time average is defined in (2). The bottom panels display the timeslot evolution of total number of packets for the same Poisson parameters.

Long-term average performance of the two policies are compared in Fig. 5 as a function of the arrival rate $\lambda$ growing from 1 to 10 in unit steps. The average is taken on the last 40000 slots, when the system runs for 50000 slots starting from zero initial condition. For $\lambda = 1$, average total number of packets under our protocol is only 312K packets, compared with 29400K packets under the V-parameter BP; likewise, the Dirichlet routing cost under our protocol is only 5100K units, compared with 91000K units under the V-parameter BP. This difference in performance gets even larger by the growth of $\lambda$.

Besides the long-term average performance in both queue congestion and routing cost, the top panels of Fig. 4 clearly show much smaller steady-state oscillations, and much faster transient-time response in the network under

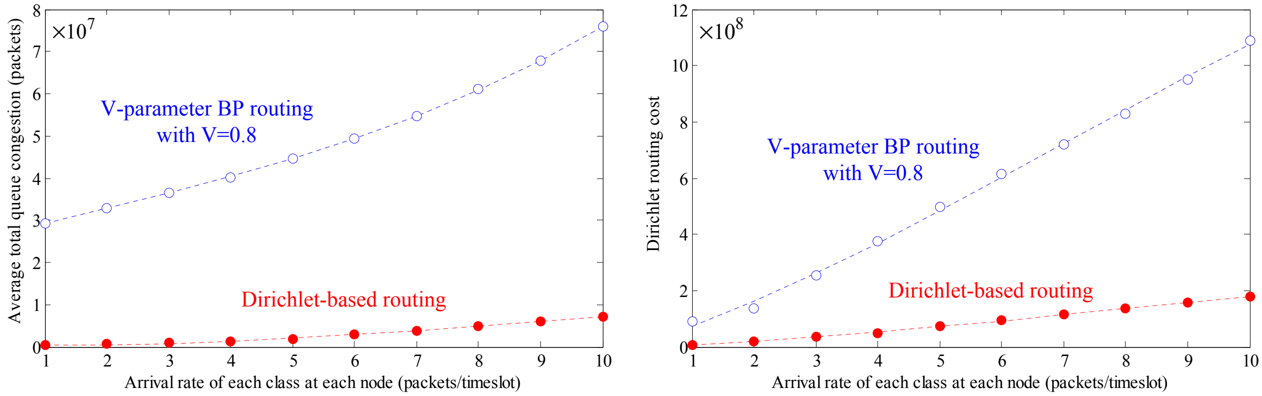


**Figure 5: Expected time average total number of packets in the network against the exogenous arrival rates changing from $\lambda = 1$ to $\lambda = 10$. Dashed lines display third degree polynomial interpolation.**

our routing protocol. Further, while our routing protocol acts immediately to the traffic rate, the V-parameter BP waits until the network reaches a minimum total queue congestion, which is bigger than 8000K packets for $V = 0.8$, while this *dead-band* grows with the increase of $V$.

## 7. CONCLUSION

We developed a network-layer routing protocol for multiclass multihop wireless networks that minimizes a general quadratic routing penalty, called the Dirichlet routing cost, subject to throughput optimality. The protocol acts dynamically with no prescribed routing path information, and requires no knowledge of arrival statistics and topology probabilities, which make it useful on time-varying mobile and ad-hoc networks. We also showed that the long-term average behavior of the stochastic wireless network under our routing protocol complies with the flow of electrical charge on its underlying graph. In doing so, we fully developed a novel concept of multiclass electric conduction on a resistive network with capacity-constrained directed edges, which brings stochastic interfering wireless and deterministic electrical networking together.

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