# Control of Open Quantum Systems in a Bosonic Bath*

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***Abstract*— We consider the *Lindblad-Kossakowski quantum master equation* describing the dynamics of an open quantum systems in the form originally proposed by Davies and Spohn. This equation contains dissipative corrections, accounting for the interaction with the environment, whose expression strongly depends on the adopted Markov approximation. In the case where a control is present, the rigorous derivation of the Markov approximation in the standard case, the *weak coupling limit*, shows that the control appears not only in the coherent part of the equation but also in the dissipative correction. This complicates the analysis of the dynamics but also offers the opportunity of indirectly affecting the interaction with the environment through the control. In this paper we study this scenario for a finite dimensional quantum system interacting with a (Bosonic) bath of harmonic oscillators. We prove several control theoretic properties of this system and discuss how the control can be used to effectively shape the influence of the environment and obtain desired features of the dynamics.**

## I. INTRODUCTION

When dealing with the analysis of the dynamics of open quantum systems, the Markovian *Quantum Master Equation (QME)* for the system density $\rho$ is one of the main tools:

$$\dot{\rho} = [-iH, \rho] + \sum_{j,k=1}^{N^2-1} d_{j,k} \Big( V_j \rho V_k^\dagger - \frac{1}{2} \left\{ V_k^\dagger V_j, \rho \right\} \Big). \quad (1)$$

Here $N$ is the dimension of the system, $d_{j,k}$ are the entries of a positive semidefinite $(N^2-1) \times (N^2-1)$ matrix, and the operators $V_j$ are the so-called *Lindblad-Kossakowski operators [3, p. 122].* $H$ is the closed system Hamiltonian. When the control $u$ is involved, it is usually meant to modify the nominal Hamiltonian $H$ in (1). However, simply replacing $H$ with $H(u)$ in (1) and leaving the dissipative operator $V_j$ unchanged is not consistent with the rigorous derivation of the QME. In fact, the dissipative operators $V_j$ depend on the nominal Hamiltonian and therefore on the control. As pointed out in [5], without further assumptions, this dependence can be very significant. This fact, which is well known in the physics literature, seems to have been somehow overlooked in the quantum control literature, where often the dissipative correction term is simply added as an additional constant term to the Schrödinger part of the QME, $\dot{\rho} = [-iH(u), \rho]$. This fact indicates that new approaches should be followed in developing a control theory for open quantum systems.

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There are several versions of the QME (1) in the literature (see, e.g., [11] for a review and [1] and the references therein for more recent work). Of particular interest to us is the work of Davies and Davies and Spohn [7], [8], [9], which allows, under appropriate restrictions, for slowly time-varying nominal Hamiltonians.

If the control enters only the coherent Hamiltonian part of the dynamics, then its 'indirect' effect on the dissipative part has to be explicitly taken into account by examining more closely the various terms in the QME. This is the approach we follow in this paper where we present a control theoretic analysis for a system immersed in a *Bosonic bath*, i.e., a bath modeled with an infinite number of harmonic oscillators. Our analysis extends and proves new control theoretic properties as compared to what was presented in [5] where the model for a quantum bit in a Bosonic bath, the *Jaynes-Cummings model*, was discussed.

## II. MASTER EQUATION IN DAVIES FORM

### A. Preliminaries

We consider a system $S$ and a bath $B$ in a total state described by a density operator $\rho_T$ on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_B$. The QME is a differential equation for the state of the system $S$, which is $\rho_S := Tr_B(\rho_T)$. We assume that the states of the system $S$ and bath $B$ are initially uncorrelated so that $\rho_T(0) = \rho_S(0) \otimes \rho_B$ for an equilibrium state of the bath $\rho_B$, i.e., $[H_B, \rho_B] = 0$. The dynamics of the total system $S + B$ is determined by an Hamiltonian operator $H_{TOT}(t)$, given by the sum of a term $\hat{H}_S(t) \otimes \mathbf{1}$, which describes the dynamics of the system alone, the term $\mathbf{1} \otimes H_B$, which describes the dynamics of the bath alone, and finally the term $\epsilon \hat{H}_{SB}$, which describes the interaction between system and bath. We have therefore

$$H_{TOT}(t) := H_S(t) \otimes \mathbf{1} + \mathbf{1} \otimes H_B + \epsilon H_{SB}. \quad (2)$$

We remark that both $H_{SB}$ and $H_B$ are assumed to be time independent, while $H_S(t) = H_S(u(t))$ is time dependent because it contains the control action. Here, following [9], we shall assume the Lamb shift to be zero. Without loss of generality, we write the interaction Hamiltonian as $\hat{H}_{SB} := \sum_j V_{Sj} \otimes V_{Bj}$, for operators $V_{S_j}$, $V_{B_j}$ .

The dynamics of $\rho_T$ follows the Liouville-Schrödinger equation

$$\begin{aligned} \dot{\rho}_T = [-iH_{TOT}(t), \rho_T] = [-iH_S(u) \otimes \mathbf{1}, \rho_T] \\ + [-i\mathbf{1} \otimes H_B, \rho_T] + \epsilon[-iH_{SB}, \rho_T]. \end{aligned} \quad (3)$$

We use the short notations $ad_S(u)$, $ad_B$, and $ad_{SB}$ for $ad_{-iH_S^\epsilon(t) \otimes \mathbf{1}}$, $ad_{-i\mathbf{1} \otimes H_B}$, and $ad_{-iH_{SB}}$, respectively, so that

(3) can be compactly written as

$$\dot{\rho}_T = (ad_S(u(t)) + ad_B)(\rho_T) + \epsilon ad_{SB}(\rho_T). \quad (4)$$

*B. The Quantum Master Equation in Davies' form*

In order to obtain a differential equation for $\rho_S$ we could simply take the partial trace with respect to the bath $B$ on the left hand side and right hand side of (4). However, although the left hand side will give $\dot{\rho}_S$, the right hand side will give an expression which does not depend only on $\rho_S$. In order for that to happen, one has to apply appropriate (Markovian) approximations. Davies considered the evolution on a long time range so that non-Markovian effects are negligible. Accordingly, the time dependence of the nominal Hamiltonian $H_S$ and therefore of the control is assumed to be slow (adiabatic limit).

Assuming that the integral converges, consider the operator $\tilde{L}$ defined as

$$\tilde{L}(t)[\rho_S] := Tr_B\left(\int_0^{+\infty} e^{-ad_S(u(t))r} \otimes e^{-ad_B r} ad_{SB} e^{ad_S(u(t))r} \otimes e^{ad_B r} ad_{SB}[\rho_S \otimes \rho_B] dr\right) \quad (5)$$

for every $\rho_S$. The operation inside the integral is the *double commutator* of $\rho_S \otimes \rho_B$ with $-iH_{SB}$ and $-i\hat{H}_{SB}(t,r)$ defined as

$$-i\hat{H}_{SB}(t,r) := e^{iH_S(u(t))r} \otimes e^{iH_B r}(-iH_{SB})e^{-iH_S(u(t))r} \otimes e^{-iH_B r}, \quad (6)$$

i.e.,

$$e^{-ad_S(u(t))r} \otimes e^{-ad_B r} ad_{SB} e^{ad_S(u(t))r} \otimes e^{ad_B r} ad_{SB}[\rho_S \otimes \rho_B] = [-i\hat{H}_{SB}(t,r), [-iH_{SB}, \rho_S \otimes \rho_B]]. \quad (7)$$

By replacing $H_{SB}$ with $\sum_j V_{S_j} \otimes V_{B_j}$ we find the following expression of $\tilde{L}$, which is useful in practical calculations:

$$\tilde{L}(t)[\rho_S] = -\sum_{j,k} \int_0^{+\infty} \Big[Tr\Big(\rho_B V_{Bj}(r) V_{Bk}\Big)\Big(V_{Sj}(t,r)V_{Sk}\rho_S - V_{Sk}\rho_S V_{Sj}(t,r)\Big) + Tr\Big(\rho_B V_{Bk} V_{Bj}(r)\Big)\Big(\rho_S V_{Sk} V_{Sj}(t,r) - V_{Sj}(t,r)\rho_S V_{Sk}\Big)\Big] dr. \quad (8)$$

Here we have also used the notation $V_{Sj}(t,r) := e^{\hat{ad}_S(u(t))r}[V_{Sj}]$, and $V_{Bj}(r) := e^{ad_B r}[V_{Bj}]$.

*Remark 1:* Davies and Spohn's analysis [9] assumes that there exists a $\lambda > 0$ such that, for any $j$ and $k$,

$$\int_0^{\infty} Tr(\rho_B V_{Bj}(r) V_{Bk})|1 + r|^{\lambda} dr < \infty, \quad (9)$$

which implies that each term in the sum (8) is finite, in addition to the sum being finite.

Consider the vector space $iu(n_S)$ of $n_S \times n_S$ Hermitian matrices. Every operator $ad_S$ on $iu(n_S)$ has the zero eigenvalue and corresponding eigenspace. The other eigenvalues come in imaginary conjugate pairs to which there correspond two-dimensional invariant eigenspaces in $iu(n_s)$ (cf. [10]). For any eigenvalue $-i\lambda_j$, we denote by $\Pi_j$ the orthogonal projection onto its eigenspace in $gl(n_S, \mathbb{C})$. For any $k$, $-i\lambda_k = i\lambda_j$ is the corresponding imaginary conjugate eigenvalue and $\Pi_k$ is the associated orthogonal projection on $gl(n_S, \mathbb{C})$. We also consider the projections onto the eigenspaces in $iu(n_S)$ corresponding to zero eigenvalue. We write the decomposition of $ad_S$ as an operator on $gl(n, \mathbb{C})$ which leaves $iu(n_S)$ invariant

$$ad_S(u(t)) := \sum_{j=1}^{n_S^2} -i\lambda_j(t)\Pi_j(t), \quad (10)$$

where some terms in the sum could be omitted since the corresponding eigenvalue could be zero. If we assume that the control $u = u(t)$ (and therefore $H_S(t)$) is an analytic function of $t$, it can be shown that both $\Pi_j$ and $\lambda_j$ are analytic functions of $t$. We assume this to be the case in the following (cf. [13] p. 120). We also define

$$K = K(t) = -\sum_{j=1}^{n_S^2} \Pi_j(t)\Pi_j'(t), \quad (11)$$

and

$$L^{\natural}(t) := \sum_{j=1}^{n_S^2} \Pi_j(t)\tilde{L}(t)\Pi_j(t). \quad (12)$$

*Theorem 1 ([9]):* Assume that the interaction operator $\hat{H}_{SB}$ and the initial equilibrium state of the bath $\rho_B$ are such that the Lamb shift $Tr_B(\hat{H}_{SB}\mathbf{1} \otimes \rho_B)$ is zero. Assume the convergence properties of Remark 1 and consider the solution $\rho^{\epsilon}$ of the linear differential equation

$$\frac{d\rho^{\epsilon}}{dt} = \Big(\epsilon^{-2} ad_S(u(t)) + K(t) + L^{\natural}(t)\Big)[\rho^{\epsilon}(t)]. \quad (13)$$

Then, for a fixed $t_0$, $\lim_{\epsilon \to 0} \sup_{0 \le t \le t_0} \|\rho_S(t) - \rho^{\epsilon}(t)\| = 0$.

The previous theorem holds in the special case where $H_S$, and therefore the control, is constant without the assumption of zero Lamb shift [8].

The dissipative corrections $K$ and $L^{\natural}$ in (11) and (12) significantly depend on the nominal Hamiltonian and therefore the control. Moreover this dependence is quite intricate. It enters in the eigen-structure of $ad_S$ as well as in the exponent of certain operators as in (5), a situation quite different from typical mathematical models considered in control theory. The question of whether this control can be used in some way to positively influence the dynamics of open quantum system is therefore quite challenging. We shall see however in the next section that some desirable properties can be obtained for the master equation with an appropriate choice of the control. The invariance property discussed in the following subsection is useful to restrict the dimension of the state space where the control problem is set.

*C. An invariance property of the QME (13)*

The dynamics of a closed system subject to a control that is specified by the nominal Hamiltonian $H_S := H_S(u)$ is characterized by the *dynamical Lie algebra* $\mathcal{L}$, which is the Lie algebra generated by the set of skew-Hermitian operators, $\mathcal{F} := \{-iH_S(u) \,|\, u \in \mathcal{U}\}$, where $\mathcal{U}$ is the space of all possible values of the control. Consider now $i\rho_S(0)$, where $\rho_S(0)$ is the initial condition of the system $S$. The *orbit*, defined as the set of states (in fact a manifold) that can be reached by the system, is a subset of the space

$$\mathcal{V} := \bigoplus_{k=0}^{\infty} ad_{\mathcal{L}}^{k} \mathtt{span}\{i\rho_S(0)\}. \tag{14}$$

For closed systems, the existence of such an invariant subspace can be effectively used to reduce the space state where the analysis is performed. Given the dependence of the dissipative correction in (13) on the nominal Hamiltonian $H_S$, the same holds true for the Lindblad-Kossakowski equation (13), as shown in the following proposition, the proof of which is relegated to the full paper:

*Proposition 1:* The vector space $\mathcal{V}$ in (14) is invariant under the dynamics (13). ■

The goal of the next sections and the main goal of the paper is to show, on a system of physical interest, that the dependence of the control of the dissipative correction can be used to obtain desirable properties of the dynamics.

## III. OPEN QUANTUM SYSTEM IN BOSONIC BATH

A Bosonic bath is a model of the environment consisting of an infinite number of harmonic oscillators at various characteristic frequencies. We assume that the finite dimensional quantum system $S$ is coupled to the Bosonic bath through the *Jaynes-Cummings Hamiltonian* (see, e.g., [3])

$$H_{SB} = \sum_j g(\omega_j)\Big(S_- \otimes b^\dagger(\omega_j) + S_+ \otimes b(\omega_j)\Big), \tag{15}$$

where $\omega_k$ is the angular frequency of the $k$-th Bosonic mode (harmonic oscillator) and $\epsilon g(\omega_k)$ its coupling to the system. As usual, $b^\dagger(\omega_k)$ and $b(\omega_k)$ are creation and annihilation operators for the $k$-th mode satisfying

$$[b(\omega_k), b(\omega_l)] = [b^\dagger(\omega_k), b^\dagger(\omega_l)] = 0;\ [b(\omega_k), b^\dagger(\omega_l)] = \delta_{kl}, \tag{16}$$

and $S_+$, $S_-$ the rising and lowering-type operators for the system $S$. The Hamiltonian for the free bath evolution, $H_B$, is given by

$$H_B = \sum_k \omega_k \Big(b^\dagger(\omega_k)b(\omega_k) + \frac{1}{2}\Big), \tag{17}$$

while we leave free the Hamiltonian of the system $H_S = H_S(u(t))$ which contains the control. Writing the interaction Hamiltonian (15) as

$$H_{SB} := S_- \otimes B_- + S_+ \otimes B_+, \tag{18}$$

where $B_- := \sum_j g(\omega_j)b^\dagger(\omega_j)$ and $B_+ : \sum_j g(\omega_j)b(\omega_j)$, the Lamb shift $Tr_B(H_{SB}\mathbf{1}\otimes\rho_B)$ is given by

$$Tr_B(H_{SB}\mathbf{1}\otimes\rho_B) = S_- Tr(B_-\rho_B) + S_+ Tr(B_+\rho_B). \tag{19}$$

We assume that the equilibrium state of the bath is given by the vacuum state $\rho_B = |0\rangle\langle 0|$, that is, no oscillator is initially excited. With this choice, we have $Tr(\rho_B b(\omega_j)) = Tr(\rho_B b^\dagger(\omega_j)) = 0$ for all $j$, and therefore $Tr(B_-\rho_B) = Tr(B_+\rho_B) = 0$, so that the *Lamb-shift* is zero. Define $b(\omega_j, t) := e^{-ad_{B}t}b(\omega_j) := e^{iH_B t}b(\omega_j)e^{-iH_B t}$ (cf. (5)). From (16) and (17), it follows that

$$[H_B, b(\omega_j)] = -\omega_j b(\omega_j), \qquad [H_B, b^\dagger(\omega_j)] = \omega_j b^\dagger(\omega_j). \tag{20}$$

This gives

$$b(\omega_j, t) = e^{-i\omega_j t}b(\omega_j), \qquad b^\dagger(\omega_j, t) = e^{i\omega_j t}b^\dagger(\omega_j). \tag{21}$$

With a view to the application in (8), we notice the equations

$$\begin{aligned}
&Tr\Big(\rho_B b(\omega_k)b(\omega_j,t)\Big) = Tr\Big(\rho_B b(\omega_k,t)b(\omega_j)\Big) = 0,\\
&Tr\Big(\rho_B b^\dagger(\omega_k)b^\dagger(\omega_j,t)\Big) = Tr\Big(\rho_B b^\dagger(\omega_k,t)b^\dagger(\omega_j)\Big) = 0,\\
&Tr\Big(\rho_B b^\dagger(\omega_k)b(\omega_j,t)\Big) = Tr\Big(\rho_B b^\dagger(\omega_k,t)b(\omega_j)\Big) = 0,\\
&Tr\Big(\rho_B b(\omega_k,t)b^\dagger(\omega_j)\Big) = e^{-i\omega_j t}\delta_{jk},\\
&Tr\Big(\rho_B b(\omega_k)b^\dagger(\omega_j,t)\Big) = e^{i\omega_j t}\delta_{jk}.
\end{aligned}$$

In our case, in the expression (8), there are only two operators $V_{Bj}$, that is, $B_-$ and $B_+$ and we use the notation $B_-(t) := e^{iH_B t}B_- e^{-iH_B t}$, $B_+(t) := e^{iH_B t}B_+ e^{-iH_B t}$, so that, using (21),

$$B_-(t) = \sum_j g(\omega_j)e^{iH_B t}b^\dagger(\omega_j)e^{-iH_B t} = \sum_j g(\omega_j)b^\dagger(\omega_j)e^{i\omega_j t}, \tag{22}$$

$$B_+(t) = \sum_j g(\omega_j)e^{iH_B t}b(\omega_j)e^{-iH_B t} = \sum_j g(\omega_j)b(\omega_j)e^{-i\omega_j t}.$$

The *autocorrelation functions*, $Tr(\rho_B V_{Bj}(r)V_{Bk})$ and $Tr(\rho_B V_{Bk}V_{Bj}(r))$, appearing in (8) in our case are

$$h^1_{\pm,\pm}(r) := \langle 0|B_\pm B_\pm(r)|0\rangle, \quad h^2_{\pm,\pm}(r) := \langle 0|B_\pm(r)B_\pm|0\rangle, \tag{23}$$

with all the possible combinations of $+$ and $-$ where the first (second) sign in $h^{1,2}_{\pm,\pm}$ refers to the first (second) sign on the right hand side. Using formulas (22) and (23) we obtain

$$\begin{aligned}
h^1_{+,+} = h^1_{-,-} = h^1_{-,+} = 0,\\
h^1_{+,-}(r) = \sum_j g^2(\omega_j)e^{i\omega_j r},\\
h^2_{+,+} = h^2_{-,-} = h^2_{-,+} = 0,\\
h^2_{+,-}(r) = \sum_j g^2(\omega_j)e^{-i\omega_j r} = h^{1\dagger}_{+,-}.
\end{aligned} \tag{24}$$

Now recall that we are required to satisfy the condition (9) of Remar 1, which, with our notation, is equivalent to

$$\int_0^\infty h^{1,2}_{\pm,\pm}(r)|1+r|^\lambda dr < \infty. \tag{25}$$

This can be obtained by an appropriate choice of the 'density' $g(\omega_j)$. Following common practice, we consider the limit to a continuum of harmonic oscillators,

$$\sum_j \ldots \to \int_{-\infty}^{+\infty} \ldots \varrho(\omega)d\omega, \tag{26}$$

where $\varrho(\omega)$ is the density of modes with angular frequency $\omega$. The sums appearing in (24) become integrals so that

$$h^1_{+,-}(r) = \int_{-\infty}^{+\infty} g^2(\omega)\varrho(\omega)e^{i\omega r}d\omega = h^{2\dagger}_{+,-}. \quad (27)$$

Condition (25) says that the following integral must converge

$$\begin{aligned}&\int_0^{+\infty} h^1_{+,-}(r)|1+r|^\lambda dr = \\ &\int_0^{+\infty}\int_{-\infty}^{+\infty} g^2(\omega)\varrho(\omega)e^{i\omega r}d\omega|1+r|^\lambda dr.\end{aligned} \quad (28)$$

We remark that it follows from (27) that $h^1_{+,-}(r)$ is, modulo a constant coefficient depending on the definition, the inverse *Fourier transform* of the function $g^2(\omega)\varrho(\omega)$. The condition of finiteness for the integral in (28) may be, for example, satisfied if $g^2(\omega)\varrho(\omega)$ is equal to $\frac{2p}{p^2+\omega^2}$ in which case $h^1_{+,-}(r)$ (and $h^2_{+,-}(r)$) is proportional to $e^{-pr}$. With these notations, the operator $\tilde{L}$ in (8) reads as

$$\begin{aligned}\tilde{L}(t)[\rho_S] = -\int_0^{+\infty}\Big(&h^1_{+,-}(r)(\rho_S S_+ S_-(t,r) - S_-(t,r)\rho_S S_+) \\ &+h^{1\dagger}_{+,-}(r)(S_+(t,r)S_-\rho_S - S_-\rho_S S_+(t,r))\Big)\, dr,\end{aligned} \quad (29)$$

where

$$S_\pm(t,r) := e^{-iH_S(t)r}S_\pm e^{iH_S(t)r}. \quad (30)$$

This is the form of the operator $\tilde{L}$ which appears in (12). The remaining features of equation (13) depend solely on the form of the nominal Hamiltonian and therefore the control.

## IV. OPEN SYSTEM IN BOSONIC BATH CONTROL

Consider the model (13), which we have specialized to an open system in a Bosonic bath with Jaynes-Commings type interaction in the previous section. The control appears, directly or indirectly, in several forms. The expression (11) also implies that the derivative of the control plays a role in the dynamics. In order to put the equation in a simpler form, we consider some special cases of the interaction $H_{SB}$ and the nominal Hamiltonian $H_S$, i.e., the control used, and analyze how the equation (13) changes. In particular we make the following simplifying assumption:

**Assumption 1** The operators $S_-$ and $S_+$ are eigenvectors of $ad_S(u(t))$, for every $u = u(t)$, corresponding to opposite eigenvalues, that is,

$$ad_S(S_-) = -i\lambda(u)S_-, \qquad ad_S(S_+) = i\lambda(u)S_+. \quad (31)$$

**Example 1** To illustrate a physical case where this assumption is verified, consider $N$ qubits in a Bosonic bath. In this case, the interaction operators $S_+$ and $S_-$ can be taken as weighted sums

$$S_+ = \sum_{k=1}^N w_k I_k^+, \qquad S_- = \sum_{k=1}^N w_k I_k^-, \quad (32)$$

where $I_k^{+,-}$ is the tensor product of $N$, $2\times 2$, identities except in the $k$ position which is occupied by the *lowering operator* $\sigma_- := \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$ in the case $-$ and the *raising operator* $\sigma_+ := \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ in the case $+$. If the nominal Hamiltonian is taken as

$$H_S := u(t)\sum_{k=1}^N I_{kz}, \quad (33)$$

where $I_{kz}$ is the tensor product of $N$ identities except in the position $k$ which is occupied by $\sigma_z := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, then Assumption 1 is verified.

Alternatively, we can consider the nominal Hamiltonian in (33) and the interaction with the environment given by

$$S_+ = w\sigma_+\otimes\sigma_+\cdots\otimes\sigma_+, \quad S_- = w\sigma_-\otimes\sigma_-\cdots\otimes\sigma_-. \quad (34)$$

In the general situation where Assumption 1 is satisfied, the operators in (30) become

$$S_+(t,r) = e^{i\lambda(u)r}S_+, \qquad S_-(t,r) = e^{-i\lambda(u)r}S_-, \quad (35)$$

and, with these expressions, the operator $\tilde{L}$ in (29) takes the form

$$\begin{aligned}\tilde{L}[\rho_S] = -\int_0^\infty &h^1_{+,-}(r)e^{-i\lambda(u)r}(\rho_S S_+ S_- - S_-\rho_S S_+) \\ &+\bar{h}^1_{+,-}(r)e^{i\lambda(u)r}(S_+S_-\rho_S - S_-\rho_S S_+)dr.\end{aligned} \quad (36)$$

By defining

$$\alpha(u) := \int_0^\infty h^1_{+,-}(r)e^{-i\lambda(u)r}dr, \quad (37)$$

we can write $\tilde{L}[\rho_S]$ as

$$\begin{aligned}\tilde{L}[\rho_S] = -\,(&\alpha(\rho_S S_+ S_- - S_-\rho_S S_+) \\ &+\alpha^*(S_+S_-\rho_S - S_-\rho_S S_+)).\end{aligned} \quad (38)$$

### A. $N = 1$, *one qubit*

Let us consider the situation of the example above described in the special case of $N = 1$. This is the common Jaynes-Cummings model for a quantum bit. We take the nominal Hamiltonian $H_S := u\sigma_z$ so that the conditions of Assumption 1 above are satisfied. Moreover, under these assumptions, $K$ in (11) is equal to zero because the projections are independent of time and the effect of changing the control is only to change the eigenvalues of $ad_S$. With the help of (38) and (12), we calculate $L^\natural(\rho_S)$ in this case by using $\rho_S := \frac{1}{2}\mathbf{1} + z\sigma_z + a\sigma_+ + a^*\sigma_-$. Denoting by $\Pi_0$, $\Pi_z$, $\Pi_+$ and $\Pi_-$ the projections onto $\texttt{span}\{\mathbf{1}\}$, $\texttt{span}\{\sigma_z\}$, $\texttt{span}\{\sigma_+\}$, $\texttt{span}\{\sigma_-\}$, respectively, we have

$$\begin{aligned}&\Pi_0\tilde{L}\Pi_0[\rho_S] = 0, \quad \Pi_z\tilde{L}\Pi_z[\rho_S] = -(\alpha+\alpha^*)z\sigma_z, \\ &\Pi_+\tilde{L}\Pi_+[\rho_S] = -a\alpha^*\sigma_+, \quad \Pi_-\tilde{L}\Pi_-[\rho_S] = -a^*\alpha\sigma_-.\end{aligned} \quad (39)$$

From these, it follows that

$$L^\natural[\rho_S] = -z(\alpha+\alpha^*)\sigma_z - a\alpha^*\sigma_+ - a^*\alpha\sigma_-. \quad (40)$$

Moreover, with $H_S = u\sigma_z$, we have

$$\frac{1}{\epsilon^2}ad_S[\rho_S] = \frac{1}{\epsilon^2}[-iu\sigma_z, \rho_S] = \frac{2iua}{\epsilon^2}\sigma_+ - \frac{2iua^*}{\epsilon^2}\sigma_-. \quad (41)$$

Therefore, Equation (13) becomes

$$\frac{d}{dt}[\rho_S] = -(\alpha+\alpha^*)z\sigma_z + \left(\frac{2iu}{\epsilon^2} - \alpha^*\right) a\sigma_+ + \left(\frac{-2iu}{\epsilon^2} - \alpha\right) a^*\sigma_-. \tag{42}$$

In general, $\alpha$ depends on both the interaction with the Bosonic bath, $h^1_{+,-}(r)$, and the eigenvalue of $ad_S$, which contains the control $u$. In particular, it is the *Laplace transform* of $h^1_{+,-}(r)$ calculated at $i\lambda = i\lambda(u)$. In our case, $\lambda(u) = 2u$. If we assume the profile example described after formula (27), i.e., $h^1_{+,-}(r) = e^{-pr}$, we have for example

$$\alpha = \int_0^\infty e^{-pr} e^{-i2ur} dr = \frac{p-2iu}{p^2+4u^2}. \tag{43}$$

In this case, Equation (42) becomes

$$\begin{aligned}\frac{d}{dt}[\rho_S] &= -\frac{2p}{p^2+4u^2} z\sigma_z \\ &+ \left(-\frac{p}{p^2+4u^2} + i2u\left(\frac{1}{\epsilon^2} - \frac{1}{p^2+4u^2}\right)\right) a\sigma_+ \\ &+ \left(-\frac{p}{p^2+4u^2} - i2u\left(\frac{1}{\epsilon^2} - \frac{1}{p^2+4u^2}\right)\right) a^*\sigma_-.\end{aligned} \tag{44}$$

This differential equation can be used to drive the state $\rho_S$ in the desired way, although, as we had said before, the control appears in a highly nonlinear way.

Consider for instance the *purity* of the state $\rho_S$ defined as

$$P := P(t) := Tr(\rho_S^2(t)) = \frac{1}{2} + 2z^2 + 2|a|^2, \tag{45}$$

which takes values between $\frac{1}{2}$ for a maximally mixed state and 1 for a pure state. Then taking the derivative of (45) and using (44) we obtain

$$\dot{P} = \frac{-2p}{p^2+4u^2(t)} P, \tag{46}$$

which describes how the purity is influenced by the control. As expected, the purity decreases with time *but this decrease can be mitigated with high amplitude control.*

### B. $N = 2$, *two qubits*

Consider now the situation described in Example 1 with the number of qubits $N$ equal to 2 and consider the interaction Hamiltonian given by $S_+$ and $S_-$ in (34) with $w = 1$ and we slightly extend the form of the nominal Hamiltonian (33) by allowing potentially independent controls $u_1$ and $u_2$ on the two qubits, i.e.,

$$H_S := u_1 \sigma_z \otimes \mathbf{1} + u_2 \mathbf{1} \otimes \sigma_z. \tag{47}$$

The eigenspaces of $-iH_S$ are $\texttt{span}\{\mathbf{1}\otimes\mathbf{1}, \sigma_z\otimes\sigma_z, \mathbf{1}\otimes\sigma_z, \sigma_z\otimes\mathbf{1}\}$, with eigenvalue zero, $\texttt{span}\{\sigma_+\otimes\mathbf{1}, \sigma_+\otimes\sigma_z\}$ with eigenvalue $-2iu_1$, $\texttt{span}\{\mathbf{1}\otimes\sigma_+, \sigma_z\otimes\sigma_+\}$, with eigenvalue equal to $-2iu_2$, $\texttt{span}\{\sigma_+\otimes\sigma_+\}$ with eigenvalue equal to $-2i(u_1+u_2)$, $\texttt{span}\{\sigma_-\otimes\sigma_-\}$ with eigenvalue equal to $2i(u_1+u_2)$, $\texttt{span}\{\sigma_-\otimes\mathbf{1}, \sigma_-\otimes\sigma_z\}$ with eigenvalue $2iu_1$, $\texttt{span}\{\mathbf{1}\otimes\sigma_-, \sigma_z\otimes\sigma_-\}$ with eigenvalue $2iu_2$, $\texttt{span}\{\sigma_+\otimes\sigma_-\}$ with eigenvalue $-2i(u_1-u_2)$, and $\texttt{span}\{\sigma_-\otimes\sigma_+\}$ with eigenvalue $2i(u_1-u_2)$.

The eigenspaces do not depend on the value of the control and therefore they do not depend on time. Therefore, in this case also, the projections $\Pi_j$ appearing in (12) are constant and $K$ in (11) is zero. We denote by $\Pi_{j,k}$ the projection onto $\sigma_j\otimes\sigma_k$, for $j,k = 0, z, +, -$, with $\sigma_0 = \mathbf{1}$, so that $L^\natural := \sum_{j,k} \Pi_{j,k}\tilde{L}\Pi_{j,k}$ in (12). From (38), denoting by $M$ the superoperator $M(\rho_S) := \rho_S S_+ S_- - S_-\rho_S S_+$, we can write

$$\tilde{L}[\rho_S] = -\alpha(u)M(\rho_S) - \alpha^*(u)M^\dagger(\rho_S), \tag{48}$$

so that

$$\begin{aligned}L^\natural[\rho_S] &= -\alpha(u)\sum_{j,k}\Pi_{j,k}M\Pi_{j,k}(\rho_S) \\ &- \alpha^*(u)\sum_{j,k}\Pi_{j,k}M^\dagger\Pi_{j,k}(\rho_S).\end{aligned} \tag{49}$$

Define $\rho_S$ as

$$\rho_S := \sum_{j,k} x_{j,k}\sigma_j\otimes\sigma_k. \tag{50}$$

We obtain the following for $\Pi_{j,k}M\Pi_{j,k}(\rho_S)$ to be used in (49):

$$\begin{aligned}\Pi_{0,0}M\Pi_{0,0}(\rho_S) = 0, \Pi_{z,z}M\Pi_{z,z}(\rho_S) &= 0,\\ \Pi_{0,z}M\Pi_{0,z}(\rho_S) &= \frac{1}{2}x_{0,z}(\mathbf{1}\otimes\sigma_z),\\ \Pi_{z,0}M\Pi_{z,0}(\rho_S) &= \frac{1}{2}x_{z,0}(\sigma_z\otimes\mathbf{1}),\\ \Pi_{-,0}M\Pi_{-,0}(\rho_S) &= \frac{1}{2}x_{-,0}(\sigma_-\otimes\mathbf{1}),\\ \Pi_{0,-}M\Pi_{0,-}(\rho_S) &= \frac{1}{2}x_{0,-}(\mathbf{1}\otimes\sigma_-),\\ \Pi_{-,-}M\Pi_{-,-}(\rho_S) &= x_{-,-}(\sigma_-\otimes\sigma_-),\\ \Pi_{-,z}M\Pi_{-,z}(\rho_S) &= \frac{x_{-,z}}{2}(\sigma_-\otimes\sigma_z),\\ \Pi_{z,-}M\Pi_{z,-}(\rho_S) &= \frac{x_{z,-}}{2}(\sigma_z\otimes\sigma_-),\end{aligned}$$

and

$$\Pi_{j,k}M\Pi_{j,k} = 0, \texttt{if } j \texttt{ or } k \texttt{ is equal to } +. \tag{51}$$

Using these in (49), we obtain

$$\begin{aligned}L^\natural[\rho_S] &= -\frac{1}{2}(\alpha+\alpha^*)x_{0,z}(\mathbf{1}\otimes\sigma_z) - \frac{1}{2}(\alpha+\alpha^*)x_{z,0}(\sigma_{\mathbf{z}}\otimes\mathbf{1})\\ &- \frac{1}{2}(\alpha x_{-,0}\sigma_-\otimes\mathbf{1} + \alpha^* x_{+,0}\sigma_+\otimes\mathbf{1})\\ &- \frac{1}{2}(\alpha x_{0,-}\mathbf{1}\otimes\sigma_- + \alpha^* x_{0,+}\mathbf{1}\otimes\sigma_+)\\ &- (\alpha x_{-,-}\sigma_-\otimes\sigma_- + \alpha^* x_{+,+}\sigma_+\otimes\sigma_+)\\ &- \frac{1}{2}(\alpha x_{-,z}\sigma_-\otimes\sigma_z + \alpha^* x_{+,z}\sigma_+\otimes\sigma_z)\\ &- \frac{1}{2}(\alpha x_{z,-}\sigma_z\otimes\sigma_- + \alpha^* x_{z,+}\sigma_z\otimes\sigma_+).\end{aligned} \tag{52}$$

To calculate $\frac{1}{\epsilon^2}ad_S(\rho_S)$, we use the formula (47) for the nominal Hamiltonian $H_S$ and the description of the eigenstructure of $ad_S$ that follows that formula. We obtain

$$\frac{1}{\epsilon^2}ad_S[\rho_S] = \sum_{j,k} \frac{x_{j,k}}{\epsilon^2} ad_S(\sigma_j \otimes \sigma_k) = \frac{-2iu_1}{\epsilon^2} x_{+,0}\sigma_+ \otimes \mathbf{1}$$
$$- \frac{2iu_1}{\epsilon^2} x_{+,z}\sigma_+ \otimes \sigma_z - \frac{2iu_2}{\epsilon^2} x_{0,+}\mathbf{1} \otimes \sigma_+$$
$$- \frac{2iu_2}{\epsilon^2} x_{z,+}\sigma_z \otimes \sigma_+ - -\frac{2i(u_1+u_2)}{\epsilon^2} x_{+,+}\sigma_+ \otimes \sigma_+$$
$$+ \frac{2i(u_1+u_2)}{\epsilon^2} x_{-,-}\sigma_- \otimes \sigma_- + \frac{2iu_1}{\epsilon^2} x_{-,0}\sigma_- \otimes \mathbf{1}$$
$$+ \frac{2iu_1}{\epsilon^2} x_{-,z}\sigma_- \otimes \sigma_z + \frac{2iu_2}{\epsilon^2} x_{0,-}\mathbf{1} \otimes \sigma_-$$
$$+ \frac{2iu_2}{\epsilon^2} x_{z,-}\sigma_z \otimes \sigma_- - \frac{2i(u_1-u_2)}{\epsilon^2} x_{+,-}\sigma_+ \otimes \sigma_-$$
$$+ \frac{2i(u_1-u_2)}{\epsilon^2} x_{-,+}\sigma_- \otimes \sigma_+.$$

Replacing this and (52) in (13), we obtain the desired controlled differential equation for $\rho_S$.

Following Prop. 1, let us assume that the initial state is an invariant subspace for $ad_S$, which Prop. 1 tells us will remain invariant during the evolution. Assume for instance that the initial state is the maximally entangled Bell state [14],

$$\begin{aligned}\rho_0 &= \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{pmatrix}\\ &= \frac{1}{2}(\sigma_+ \otimes \sigma_+ + \sigma_- \otimes \sigma_-) + \frac{1}{4}(\mathbf{1} \otimes \mathbf{1} + \sigma_z \otimes \sigma_z).\end{aligned} \tag{53}$$

Since $\mathtt{span}\{\mathbf{1}\otimes\mathbf{1}, \sigma_z\otimes\sigma_z, \sigma_+\otimes\sigma_+, \sigma_-\otimes\sigma_-\}$ is invariant, equation (13) simplifies to

$$\dot{x}_{0,0} = 0, \quad \dot{x}_{z,z} = 0 \tag{54}$$
$$\dot{x}_{-,-} = \left[-\alpha(u) + i\frac{2(u_1+u_2)}{\epsilon^2}\right] x_{-,-},$$
$$\dot{x}_{+,+} = \left[-\alpha^\dagger(u) - i\frac{2(u_1+u_2)}{\epsilon^2}\right] x_{+,+},$$

where the last equation is redundant since $x_{+,+} = x^\dagger_{-,-}$, and all other derivatives and components are zero. By adapting what done in (43) we get

$$\alpha = \frac{p - 2i(u_1+u_2)}{p^2 + 4(u_1+u_2)^2}. \tag{55}$$

By writing $x_{-,-}$ as $x_{-,-} := x + iy$, and writing $v := -2(u_1+u_2)$, the last equation of (54) is written as

$$\dot{x} = -\frac{p}{p^2+v^2}x + v\left(\frac{1}{p^2+v^2} + \frac{1}{\epsilon^2}\right)y, \tag{56}$$

$$\dot{y} = -\frac{p}{p^2+v^2}y - v\left(\frac{1}{p^2+v^2} + \frac{1}{\epsilon^2}\right)x. \tag{57}$$

If we use the as the measure of entanglement the *concurrence* (see, e.g., [14]), then we have that the amount of entanglement of the state $\rho_S$ is given by $C(\rho_S) := 2\sqrt{x^2+y^2}$, which decays exponentially according to the real part of the eigenvalues of the system in (56)-(57) if $v$ is constant, that is, according to $-\frac{p}{p^2+v^2}$.

## V. CONCLUDING REMARKS

When using the Quantum Master Equation (QME) as a model for the control of open quantum systems, the control appears not only in the nominal Hamiltonian but also, indirectly, in the dissipative correction. On one hand, it complicates the analysis, but on the other hand it offers opportunities for control design. As for closed systems, Davies' QME preserves the invariance of certain subspaces.

As an illustration of the control theoretic features of the QME in Davies form, we have analyzed the 'controlled' QME for a Jaynes-Cummings model of one and two qubits in interaction with a Bosonic bath. When the control goals are purity and entanglement, it appears that the only prescription to improve these features is to use high amplitude control. Moreover, in general, there are features of the dynamics that lead to a decay of the state towards the perfectly mixed state and are essentially independent of the control used and cannot be eliminated. Nevertheless, as one of the major results of modeling the effect of the control on the dissipative part of the Lindblad equation, the rate of decay can be mitigated by control. This is quite an improvement over the simple coherent control that has no effect on decoherence [12]. Furthermore, the models can be used for a direct controllability analysis when one tries to identify the available states and/or a way to drive between two states.


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