Robust Control Performance for Open Quantum Systems

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Abstract

The robustness of quantum control in the presence of uncertainties is important for practical applications but their quantum nature poses many challenges for traditional robust control. In addition to uncertainties in the system and control Hamiltonians and initial state preparation, there is uncertainty about interactions with the environment leading to decoherence. This paper investigates the robust performance of control schemes for open quantum systems subject to such uncertainties. A general formalism is developed, where performance is measured based on the transmission of a dynamic perturbation or initial state preparation error to a final density operator error. This formulation makes it possible to apply tools from classical robust control, especially structured singular value analysis, to assess robust performance of controlled, open quantum systems. However, there are additional difficulties that must be overcome, especially at low frequency ($\mathbf{s} \approx \mathbf{0}$). For example, at $\mathbf{s} = \mathbf{0}$, the Bloch equations for the density operator are singular, and this causes lack of continuity of the structured singular value. We address this issue by analyzing the dynamics on invariant subspaces and defining a pseudo-inverse that enables us to formulate a specialized version of the matrix inversion lemma. The concepts are demonstrated with an example of two qubits in a leaky cavity under laser driving fields and

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spontaneous emission. In addition, a new performance index is introduced for this system. Instead of the tracking or transfer fidelity error, performance is measured by the steady-steady entanglement generated, which is quantified by a non-linear function of the system state called concurrence. Simulations show that there is no conflict between this performance index, its log-sensitivity and stability margin under decoherence, unlike for conventional control problems where a trade-off between the tracking error and its log-sensitivity usually exists.

Index Terms

Quantum information and control, uncertain systems, robust control, H-infinity control.

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I. Introduction

Quantum control offers a variety of techniques to steer the dynamics of quantum systems. This is essential to enable a wide range of applications for quantum technologies. However, uncertainties arising from limited knowledge of Hamiltonians, decoherence processes and initial state preparation errors alter quantum dynamics and the effectiveness of the control schemes. While classical robust control has developed effective solutions for such situations, they do not apply straightforwardly to quantum control. To consider the robustness of quantum control strategies in the presence of uncertainties, we develop a general formalism where the performance is the δ-strength structured uncertain transmission $T_{\boldsymbol{z},\boldsymbol{w}}(s,\delta)$ (of the usual Laplace frequency parameter s), from the generalized "noise" $\boldsymbol{w}$ (e.g., decoherence, preparation errors) to the density operator error $\boldsymbol{z}$. It is tacitly assumed that this response has been made "small" by the control design under nominal values of the Hamiltonian parameters and decoherence ($\delta = 0$). Robust performance is hence defined as the ability of $T_{\boldsymbol{z},\boldsymbol{w}}(s,\delta)$ to remain within identifiable bounds when $\delta \neq 0$. Since uncertainties in Hamiltonians and, to a lesser extent, Lindbladian decoherence are *structured* with strength δ, it is natural to quantify robustness of the performance using structured singular values. A generic difficulty in dealing with quantum systems is that the constancy of the trace of the density matrix ρ imposes a pole at $s = 0$ in the $T_{\boldsymbol{z},\boldsymbol{w}}(s,\delta)$ dynamics. This creates a singularity of the dynamics at low frequencies, $s \approx 0$, mandating some revision of the traditional machinery of the structured singular value and a specially dedicated *matrix #-inversion lemma*, not to be confused with the matrix pseudo-inversion lemma [1], [2], is required. This singularity situation is reminiscent of deliberately adding an integrator along

the open-loop dynamics, and hence a pole at $s = 0$, to ensure the asymptotic tracking error vanishes [3].

After reviewing quantum dynamics and specific uncertainties in Sec. II, we introduce in Sec. III a novel, general formalism that reduces robustness against *all* uncertainties to enforcing the *single* transmission $T_{\boldsymbol{z},\boldsymbol{w}}$ to be robust against Hamiltonian parameter uncertainties [4] and decoherence strength [5]. Preparation error response requires a different formulation departing from classical robust performance, as treated in [6], [7]. In Sec. IV the case of pure dephasing in the Hamiltonian basis is developed and analytic bounds for the error transmission $T_{\boldsymbol{z},\boldsymbol{w}}$ are derived. Sec. V addresses generic dissipation and develops a generalized framework to deal with the $s = 0$ singularity. Robust performance for generic dissipative dynamics is illustrated by a case study of two qubits in a cavity in Sec. VI. While this is a simple example, it allows the formulation of another innovation in robust control in that it considers a case where the performance measure is given by the concurrence, a measure of entanglement between the two qubits in the cavity, which is a *nonlinear* function of the system state (Sec. VI-E). Although the cavity is dissipative, it shows a lack of conflict between the concurrence and its log-sensitivity [8], while other pairs of performance measures still show classical conflicts. For convenience, Table I summarises the notation used in this paper.

II. Quantum Dynamics and Uncertainties

We briefly review quantum dynamics with uncertainties to set up the basic formalism of our approach.

A. Schrödinger and Liouville Equations

The dynamics of a quantum system, whose pure states $|\Psi(t)\rangle$ are wavefunctions in a Hilbert space $\mathbf{H} = \mathbb{C}^N$, are typically described by the time-dependent Schrödinger equation, $\frac{d}{dt}|\Psi(t)\rangle = -\imath H|\Psi(t)\rangle$ (in a system of units where the reduced Planck Constant $\hbar = 1$), or the quantum Liouville equation for density operators ρ,

$$\tfrac{d}{dt}\rho(t) = -\imath[H, \rho(t)], \tag{1}$$

where H is the Hamiltonian of the system, ρ is a (bounded) Hermitian operator on $\mathbf{H}$ with $\mathrm{Tr}(\rho) = 1$ and $[A, B] = AB - BA$ is the usual matrix commutator. For pure states, the density operator is simply the projector onto $|\Psi\rangle$, i.e., $\rho = |\Psi\rangle\langle\Psi|$. If $\dim \mathbf{H} = N < \infty$, H and ρ can be

TABLE I: Notation

$\mathbf{H}$	Hilbert space of quantum system $\simeq \mathbb{C}^N$.
$\mathbf{Herm}$	Hermitian operators $\mathbf{H} \to \mathbf{H}$.
ρ	$N \times N$ Hermitian density operator of Trace 1.
H	$N \times N$ Hermitian Hamiltonian operator.
$\mathfrak{L}(V)$	$N \times N$ Lindbladian with jump operator(s) V (Eq. (2)).
δ	Strength of decoherence or strength of general uncertainty.
$\boldsymbol{r}$	N^2-dimensional Block representation vector of ρ.
$\boldsymbol{z}$	N^2-dimensional Bloch representation vector of error on ρ.
$\boldsymbol{w}$	Extraneous disturbance (noise or preparation error).
$T_{\boldsymbol{z},\boldsymbol{w}}$	Transmission from $\boldsymbol{w}$ to $\boldsymbol{z}$.
A, S	$N^2 \times N^2$ Bloch representations of nominal H and nominal $\mathfrak{L}(V)$, resp. (Eq. (5)).
$\Phi(s)$	$sI - A$.
$\mathcal{S}$	$N^2 \times N^2$ Bloch representation of disturbance on A for enhanced uncertainty structure (Eqs. (47)-(48)).
$\sigma_x, \sigma_y, \sigma_z$	Pauli operators (Eq. (42)).

represented by $N \times N$ Hermitian matrices. The advantage of density operators and the quantum Liouville equation is that they can easily be extended to describe open system dynamics by adding a Lindbladian term,

$$\mathfrak{L}(V)\rho = V\rho V^\dagger - \tfrac{1}{2}(V^\dagger V\rho + \rho V^\dagger V), \tag{2}$$

to the right-hand side of Eq. (1), resulting in the Liouville-Lindblad master equation

$$\tfrac{d}{dt}\rho(t) = -\imath[H, \rho(t)] + \delta\mathfrak{L}(V)\rho(t), \tag{3}$$

where δ is the decoherence rate, which can be interpreted as the *strength* of the *structured* perturbation defined by $\mathfrak{L}(V)$.

B. Initial State Preparation Errors versus Dephasing

When studying quantum dynamics subject to a variety of structured perturbations, it is important to note that some of these perturbations may be indistinguishable, in particular, initial state preparation error and decoherence. If we measure the difference between the actual resulting state $\tilde{\rho}$ and a desired state ρ, we cannot determine if the error at a particular time t is the result of dynamic dephasing of a perfectly prepared pure initial state, a mixed initial state evolving

perfectly according to Hamiltonian dynamics, or a combination of both. This is illustrated with a simple example that also serves as a "warm-up exercise."

Example 1: Consider a two-level system with $|0\rangle = (1,0)^T$ and $|1\rangle = (0,1)^T$. Assume the system is prepared in a pure state $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and evolves under the diagonal Hamiltonian $H = \omega\sigma_z$ with $\sigma_z = \mathrm{diag}(-1,1)$ while dephasing $V = \sigma_z$ acts in the Hamiltonian basis ($[H,V] = 0$) at a rate δ. In this case equation (3) gives

$$\rho_0 = \rho(0) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \rho(t) = \frac{1}{2}\begin{bmatrix} e^{2\imath\omega t} & e^{-2\delta t} \\ e^{-2\delta t} & e^{-2\imath\omega t} \end{bmatrix}.$$

Alternatively, if we initially prepare a mixed state $\tilde{\rho}_0$ that evolves under the same Hamiltonian $H = \omega\sigma_z$ without dephasing then we obtain

$$\tilde{\rho}_0 = \rho(0) = \frac{1}{2}\begin{bmatrix} 1 & e^{-\tau\delta} \\ e^{-\tau\delta} & 1 \end{bmatrix}, \quad \tilde{\rho}(t) = \frac{1}{2}\begin{bmatrix} e^{2\imath\omega t} & e^{-\tau\delta} \\ e^{-\tau\delta} & e^{-2\imath\omega t} \end{bmatrix}.$$

Clearly, the two solutions are indistinguishable at $t = \tau/2$.

C. Genericity and Stability of Hamiltonian Eigenstructure

To be able to develop a general formalism and prove certain results we make some assumptions about genericity of the eigenstructure of the Hamiltonian, and whether multiple eigenvalues, if any, will survive under physical parameter drift. Clearly changing multiplicity of the eigenvalues may give rise to unstable dynamics. Specifically, consider a quantum system with Hilbert-space dimension N and let $n_k = \mathrm{rank}(\Pi_k)$, where $\{\Pi_k\}_{k=1}^{\bar{N}\leq N}$ is the family of projectors onto the respective eigenspaces of the Hamiltonian associated with the eigenvalues $\lambda_k(H)$ with $\sum_{k=1}^{\bar{N}} n_k = N$. Such an eigenstructure is *in general* unstable under perturbation in the sense that eigenvalues with multiplicity greater than 1 will split into several lower multiplicity eigenvalues under universal unfolding [9]. Securing stability of the eigenstructure of H (and V) requires $\bar{N} = N$, an assumption that can be justified invoking *genericity*. Such issue specialized to energy landscape traces back to von Neumann and Wigner [10] and was further developed in [11], [12].

Definition 1: A property $\mathfrak{P}$ of a set of matrices $\mathcal{M}$ is said to be *generic* if the subset of matrices where $\mathfrak{P}$ holds is open and dense in $\mathcal{M}$ for a relevant topology on $\mathcal{M}$.

Theorem 1: For $N \geq 2$, the subset $\mathfrak{M}_{n_1,n_2,\ldots,n_{\bar{N}}} \subset \mathcal{H}(N)$ of the set of $N \times N$ Hermitian matrices with eigenvalues with multiplicities $n_1, n_2, \ldots, n_{\bar{N}}$ is a $\mathbb{R}^*$-homogeneous sub-manifold of codimension $\left(\sum_{k=1}^{\bar{N}} n_k^2\right) - \bar{N}$ in $\mathcal{H}(N)$. Moreover, the subset $\mathfrak{V} \subset \mathcal{H}(N)$ of the set of $N \times N$

Hermitian matrices with multiple eigenvalues is a real algebraic variety $\mathfrak{V}$ of codimension 3 in $\mathcal{H}(N)$.

See [11, Corollary 4.12].

Corollary 1: The property $\mathfrak{P}$ of "no multiple eigenvalues" is generic in $\mathcal{H}(N)$.

Corollary 2: In an Hermitian, continuously real, p-parameterized family H_p, embedded in $\mathcal{H}(N)$ with $H_0 \in \mathcal{H}(N) \setminus \mathfrak{V}$ (that is, H_0 has no multiple eigenvalues), generically, a three-dimensional (real) perturbation $p = \Big(p_1, p_2, p_3\Big)$ is necessary to reach multiple eigenvalues. Under nongeneric conditions, more than three parameters are needed, unless there exists a unique k^* such that $n_{k^*} = 2$ and $n_{k\neq k^*} = 1$ in which case three parameters are still enough.

See Appendix.

Practically speaking, from the proof in the Appendix, the argument can be reversed to arbitrarily split a multiple eigenvalue into several simple eigenvalues under the $\mathfrak{V}$-generic condition that 3 uncertain parameters are enough. The simplest system of two coupled qubits already has 16 parameters in its 4×4 Hamiltonian, all of which are uncertain to some degree. If we consider the simplest case of one and only one double eigenvalue, we have codim $\left(\mathfrak{M}_{n_1=2,n_2=n_3=1}\right) = 3$. Therefore, the double eigenvalue can be arbitrarily split with 3 parameters. Note, however, that any higher multiplicity structure of the eigenvalues (precisely, $\bar{N} < 3$) would create codim $\left(\mathfrak{M}_{n_1,n_2,\ldots,n_{\bar{N}}}\right) > 3$ and therefore more than 3 parameters would be needed to achieve an arbitrary splitting. To put it simply, given the high number of uncertain parameters in quantum systems, the "no multiple eigenvalues" assumption is reasonable.

Remark 1: An early version of Corollary 2 is in [10]. The difference is that, here, we have clarified what is meant by "in general" (*"im allgemeinen"*) on [10, p. 553] while referring to the sufficiency of 3 parameters. Precisely, across $\mathfrak{V}$ the most "general" singularities are those of the smallest codimension, which happens to be 3.

D. Bloch Equation

To simplify the analysis we reformulate the quantum dynamics Eq. (3) as a linear ODE for a real state vector $\boldsymbol{r}$,

$$\tfrac{d}{dt}\boldsymbol{r}(t) = (A + \delta S)\boldsymbol{r}(t), \tag{4}$$

by expanding ρ and the Hermitian dynamical generator(s) with respect to a suitable basis for the operators on the Hilbert space $\mathbf{H}$, e.g., the generalized Pauli or Gell-Mann basis [13]. For

example, if $\{\boldsymbol{\sigma}_n\}_{n=1}^{N^2}$ is an orthonormal basis for the $N\times N$ Hermitian matrices with $\boldsymbol{\sigma}_{N^2}=\frac{1}{\sqrt{N}}I$ then $\boldsymbol{r}=(r_n)_{n=1}^{N^2}$ with $r_n=\mathrm{Tr}(\boldsymbol{\sigma}_n\rho)$. This leads to

$$A_{mn}=\mathrm{Tr}(\imath H\boldsymbol{\sigma}_m,\sigma_n]), \tag{5a}$$

$$S_{mn}=\mathrm{Tr}(V\boldsymbol{\sigma}_m V\sigma_n-\tfrac{1}{2}V^2\{\boldsymbol{\sigma}_m,\boldsymbol{\sigma}_n\}). \tag{5b}$$

Observe that A is real, anti-symmetric, while S is symmetric. From $r_{N^2}=\mathrm{Tr}\left(\frac{1}{\sqrt{N}}\rho\right)=\frac{1}{\sqrt{N}}$, it follows that $\dot{r}_{N^2}(t)\equiv 0$. So the last row of $A+\delta S$ vanishes for all δ. Thus, in general $\mathrm{rank}(A+\delta S)\leq N^2-1$, $\forall\delta$. Moreover, $\mathrm{rank}(A)\leq N^2-1$ and $\mathrm{rank}(S)\leq N^2-1$, separately, as this rank deficiency is a consequence of the choice of the basis operator $\boldsymbol{\sigma}_{N^2}$ and independent of the dynamical generators.

E. Control and State Feedback

We are concerned with the performance of controlled quantum systems. The most natural formulation of quantum control is a control-dependent Hamiltonian $H=H[\boldsymbol{f}]$, leading to a non-linear control system. In the simplest case, where $H[\boldsymbol{f}]=H_0+\sum_{m=1}^{M}f_m(t)H_m$ is linear in the controls $f_m(t)$, we obtain a bilinear control system with the Bloch equation

$$\dot{\boldsymbol{r}}(t)=\left[A_0+\sum_{m=1}^{M}f_m(t)A_m\right]\boldsymbol{r}(t), \tag{6}$$

where $A_0,(A_1,\ldots,A_M)$ are $N^2\times N^2$ real matrices describing the system and control dynamics, respectively. However, the dynamics can be equivalently written as a linear system,

$$\dot{\boldsymbol{r}}(t)=A\boldsymbol{r}(t)+B\boldsymbol{u}(t), \tag{7}$$

if we set $A=A_0$, $B=I$ and define the *state-feedback*

$$\boldsymbol{u}(\boldsymbol{r},t)=\left[\sum_{m=1}^{M}f_m(t)A_m\right]\boldsymbol{r}(t). \tag{8}$$

If the $f_m(t)$ are time-independent controls, the resulting system is a linear time-invariant (LTI) system with autonomous state feedback [4]. Note, however, that stabilization of such systems by choosing some f_m to produce specified eigenvalues for $A+\sum_m f_m A_m$ does not reduce to the well-known pole placement method; the latter requires feedback of the form $\boldsymbol{u}(t)=K\boldsymbol{r}(t)$ with K completely free except for its size, whereas here K is constrained to be $\sum_m f_m A_m$ [14].

Note that $\sum_{m=1}^{M}f_mA_m$ can also model system errors, with A_m being the structure of the uncertainty and f_m its strength. We will take this path of approach in the following section.

III. Robust Performance in Open Quantum Systems

Performance of a controlled quantum system under uncertainties can be characterized by a transmission function $T_{\boldsymbol{z},\boldsymbol{w}}$. To rigorously develop this framework, consider an unperturbed and a perturbed system with state vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_p$, respectively, evolving according to

$$\dot{\boldsymbol{r}}_u(t) = A\boldsymbol{r}_u(t), \tag{9a}$$

$$\dot{\boldsymbol{r}}_p(t) = (A + \delta S)\boldsymbol{r}_p(t), \tag{9b}$$

where A describes dynamics of the ideal unperturbed system and δS is a perturbation to the dynamics of structure S and magnitude δ. The resulting error vector $\boldsymbol{z}(t) = \boldsymbol{r}_p(t) - \boldsymbol{r}_u(t)$ satisfies either of the dynamics

$$\dot{\boldsymbol{z}}(t) = (A + \delta S)\boldsymbol{z}(t) + \delta S\boldsymbol{w}_u(t), \tag{10a}$$

$$\dot{\boldsymbol{z}}(t) = A\boldsymbol{z}(t) + \delta S\boldsymbol{w}_p(t), \tag{10b}$$

with feedback $\boldsymbol{w}_u(t) = \boldsymbol{r}_u(t)$ and $\boldsymbol{w}_p(t) = \boldsymbol{r}_p(t)$, respectively.

A case can be made for Eq. (10b), as the error $\boldsymbol{z}(t)$ is the "noise response" to a *perturbed* signal and hence the formulation is in line with the "disturbance rejection" paradigm. But a case can also be made for Eq. (10a) as the error dynamics are the real perturbed dynamics, but driven by an unperturbed purely oscillatory signal with *known* eigenfrequencies, so that the frequency sweep can be limited to finitely many known frequencies.

From another perspective, the transfer matrices of the unperturbed, perturbed cases, $T^u_{\boldsymbol{z},\boldsymbol{w}_u}$, $T^p_{\boldsymbol{z},\boldsymbol{w}_p}$, resp., are the variations of $(sI - A)^{-1}$ relative to either the unperturbed dynamics or the perturbed dynamics as a function of $\delta \geq 0$:

$$\begin{aligned}
T^u_{\boldsymbol{z},\boldsymbol{w}_u} &:= (sI - A - \delta S)^{-1}\delta S \\
&= \left[(sI - A - \delta S)^{-1} - (sI - A)^{-1}\right]\left[(sI - A)^{-1}\right]^{-1}, \\
T^p_{\boldsymbol{z},\boldsymbol{w}_p} &:= (sI - A)^{-1}\delta S \\
&= \left[(sI - A - \delta S)^{-1} - (sI - A)^{-1}\right]\left[(sI - A - \delta S)^{-1}\right]^{-1}.
\end{aligned}$$

Whether the variation of the *error transmission* matrix should be scaled relative to the inaccurate, but known model or the true, but unknown dynamics is a matter of preference, with the former perceived as more appropriate [15, Sec. II.C]. Therefore, after briefly analyzing both points of view in the next two subsections, the "unperturbed" case will be the preferred method.

A. Unperturbed State Feedback

Firstly, consider the point of view taken by Eq. (10a) of unperturbed state feedback. Taking its Laplace transform yields

$$(sI - A - \delta S)\hat{\boldsymbol{z}}(s) = \delta S\hat{\boldsymbol{w}}_u(s) + \boldsymbol{z}(0). \tag{11}$$

If $sI - A - \delta S$ is invertible and there is no initial state preparation error, $\boldsymbol{z}(0) = \mathbf{0}$, then

$$\hat{\boldsymbol{z}}(s) = (sI - A - \delta S)^{-1}\delta S\hat{\boldsymbol{w}}_u(s) = T^u_{\boldsymbol{z},\boldsymbol{w}_u}(s)\hat{\boldsymbol{w}}_u(s). \tag{12}$$

The formulation of Eq. (12) enables structured singular value analysis [6], [7]. *Assuming that* $\Phi(s) := (sI - A)$ *is invertible,* by simple matrix manipulation, we get

$$(\Phi(s) - \delta S)^{-1}\delta S = (I - \Phi(s)^{-1}S(\delta I))^{-1}\Phi(s)^{-1}S(\delta I). \tag{13}$$

The above reveals that the error response $T^u_{\boldsymbol{z},\boldsymbol{w}_u} := (sI - A - \delta S)^{-1}\delta S$ is obtained from

$$\begin{pmatrix} v \\ \hat{\boldsymbol{z}}(s) \end{pmatrix} = \underbrace{\begin{pmatrix} \Phi(s)^{-1}S & \Phi(s)^{-1}S \\ I & 0 \end{pmatrix}}_{G_{\boldsymbol{z},\boldsymbol{w}_u}(s)} \begin{pmatrix} \eta \\ \hat{\boldsymbol{w}}_u(s) \end{pmatrix}, \tag{14}$$

after the feedback $\eta = (\delta I)v$. Next, a *fictitious* feedback $\boldsymbol{w}_u = \Delta_f \boldsymbol{z}$ allows us to compute $\|T^u_{\boldsymbol{z},\boldsymbol{w}_u}(s)\| = 1/\min\{\|\Delta_f\| : \det(I + T^u_{\boldsymbol{z},\boldsymbol{w}_u}(s)\Delta_f) = 0\}$. Putting the two feedbacks together as $\boldsymbol{\Delta} = \begin{pmatrix} \delta I & 0 \\ 0 & \Delta_f \end{pmatrix}$ and using the matrix inversion lemma yields the robust performance theorem [7, Th. 10.8]:

Theorem 2: If $\Phi(s)$ is invertible, $\|T^u_{\boldsymbol{z},\boldsymbol{w}}(s)\| \leq \mu_{\mathcal{D}}(G(s))$ for $\delta < 1/\mu_{\mathcal{D}}(G(s))$, where $\mathcal{D}$ is the structure defined by the block-diagonal matrix $\mathrm{diag}(M_1, M_2)$ where M_1 is an $N^2 \times N^2$ real diagonal matrix, M_2 is an $N^2 \times N^2$ complex matrix, and

$$\mu_{\mathcal{D}}(G(s)) = \frac{1}{\min\{\|\boldsymbol{\Delta} \in \mathcal{D}\| : \det(I + G(s)\boldsymbol{\Delta}) = 0\}}$$

is the structured singular value specific to $\mathcal{D}$.

It is important to note that the representation of the uncertainty as diagonal feedback about a "certain" plant has a problem when $\Phi(s)$ is not invertible. Unfortunately, when A is the Bloch matrix of an open quantum system then it has an eigenvalue at 0 so that $\Phi(s)$ has a pole at $s = 0$, invalidating the above representation. This issue, which is often overlooked in the application of the matrix inversion lemma [1], [2], is central in the proof of Th. 2. It is examined in Secs. V-A and V-B, where a specialized "pseudo-inverse" will be introduced to deal with this singularity.

Remark 2: The authors of [3] faced a similar problem in the μ-synthesis of a PI disk drive controller. They report significant numerical difficulties at "low frequency," obviously due to the presence of the open-loop pole at $s = 0$. To obviate such difficulties, they elect to do the μ-design at "medium to high frequencies" and then add the PI component of the controller. In Secs. V-A and V-B, we address this difficulty by looking at the μ-analysis all the way down to $s = 0$.

B. Perturbed State Feedback

Next, let us briefly consider the point of view taken by Eq. (10b) of perturbed state feedback. Taking its Laplace transform yields

$$(sI - A)\hat{\boldsymbol{z}}(s) = \delta S\hat{\boldsymbol{w}}_p(s) + \boldsymbol{z}(0). \tag{15}$$

If $sI - A$ is invertible and there is no initial state preparation error, $\boldsymbol{z}(0) = \mathbf{0}$, then

$$\hat{\boldsymbol{z}}(s) = (sI - A)^{-1}\delta S\hat{\boldsymbol{w}}_p(s) = T^p_{\boldsymbol{z}\boldsymbol{w}_p}(s)\hat{\boldsymbol{w}}_p(s). \tag{16}$$

Bounding $\|T^p_{\boldsymbol{z}\boldsymbol{w}_p}(s)\|$ does not require the matrix inversion lemma nor the structured singular value analysis. This apparent simplification, however, overlooks the fact that $T^p_{\boldsymbol{z}\boldsymbol{w}_p}(s)$, perturbed by δ, is driven by a signal also perturbed by δ. Therefore, rigorous analysis of bounding $\hat{\boldsymbol{z}}(s)$ would require further work. So in what follows, we instead consider the approach where the driving signal is unperturbed.

C. Initial State Preparation Error

The previous subsections assumed that the initial state was prepared perfectly. We can also treat initial state preparation error in this framework from both the unperturbed and perturbed points of view.

Introducing $\boldsymbol{z}(0)$ in Eq. (10a) and restricting Eq. (11) to the initial state preparation error yields in the unperturbed case

$$(sI - A - \delta S)\hat{\boldsymbol{z}}(s) = \boldsymbol{z}(0). \tag{17}$$

If $sI - A - \delta S$ is invertible, this yields

$$\hat{\boldsymbol{z}}(s) = (sI - A - \delta S)^{-1}\boldsymbol{z}(0) = T^u_{\boldsymbol{z},\boldsymbol{z}_0}\boldsymbol{z}(0). \tag{18}$$

This is the path taken in [6], which required the matrix inversion lemma and the structured singular values. Precisely, if $\Phi(s)$ (defined Sec. III-A) is invertible, we have

$$(\Phi - \delta S)^{-1} = \Phi^{-1} + \Phi^{-1}(\delta I)(I - S\Phi^{-1}(\delta I))^{-1}S\Phi^{-1}.$$

It follows that $T^u_{\boldsymbol{z},\boldsymbol{z}_0}(s)$ can be obtained from

$$\begin{pmatrix} v \\ \hat{\boldsymbol{z}}(s) \end{pmatrix} = \underbrace{\begin{pmatrix} S\Phi^{-1}(s) & S\Phi^{-1}(s) \\ \Phi^{-1}(s) & \Phi^{-1}(s) \end{pmatrix}}_{G^u_{\boldsymbol{z},\boldsymbol{z}_0}(s)} \begin{pmatrix} \eta \\ \boldsymbol{z}_0 \end{pmatrix}, \tag{19}$$

after the feedback $\eta = (\delta I)v$ and moreover $\|T^u_{\boldsymbol{z},\boldsymbol{z}_0}(s)\|$ is simultaneously computed via the compound feedback $\boldsymbol{\Delta} = \begin{pmatrix} \delta I & 0 \\ 0 & \Delta_f \end{pmatrix}$. The related robust performance theorem is a straightforward adaptation of Th. 2 and is left to the reader.

In the perturbed case, introducing $\boldsymbol{z}(0)$ in the perturbed Eq. (10b), we obtain instead

$$\hat{\boldsymbol{z}}(s) = (sI - A)^{-1}\boldsymbol{z}(0) = T^p_{\boldsymbol{z},\boldsymbol{z}_0}(s)\boldsymbol{z}(0), \tag{20}$$

assuming that $sI - A$ is invertible. Bounding $\|T^p_{\boldsymbol{z},\boldsymbol{z}_0}(\imath\omega)\|$ amounts to a classical frequency sweep.

IV. Pure Dephasing in Hamiltonian Basis

In this section we apply the formalism derived in the previous section to study the performance of controlled quantum systems subject to dephasing in the Hamiltonian basis, a typically undesired behavior commonly encountered for quantum systems interacting weakly with an environment. In this special case we can assume that dephasing acts in the eigenbasis of the Hamiltonian, i.e., $[H, V] = 0$.

A. Quantum Dynamics subject to Pure Dephasing

With the added condition $[H, V] = 0$ in the Lindblad master equations (2)-(3), we can say much more than Sec. II-D about the Bloch equation. Firstly, we need the following lemmas and corollary:

Lemma 1: Let P, Q be $N \times N$ Hermitian operators on $\mathbf{H}$ that commute. Then

1) If the (orthonormalized) bases of the eigenspaces of P or Q associated with the multiple egenvalues are constrained, then P and Q are simultaneously *block* diagonalizable via a unitary transformation.

2) If the (orthonormalized) bases of the eigenspaces of P and Q associated with multiple eigenvalues are freely adjustable, then P and Q are simultaneously diagonalizable by a unitary transformation.

See Appendix.

Corollary 3: Under the same conditions as Lemma 1, the kernel of one operator equals the direct sum of selected invariant subspaces of the other. Moreover, if one such invariant subspace corresponds to an eigenvalue $\neq 0$ the kernels of P and Q are not coincidental.

See Appendix.

Lemma 2: Let P, Q be $N \times N$ Hermitian operators. If $\mathrm{Tr}(P^\dagger Q) = 0$ in the Lindblad master equation, then $((\mathrm{Tr}(P^\dagger \boldsymbol{\sigma}_n))_{n=1}^{N^2})^\dagger (\mathrm{Tr}(Q^\dagger \boldsymbol{\sigma}_n))_{n=1}^{N^2} = 0$ in the Bloch representation.

Expand P and Q in terms of the basis $\{\boldsymbol{\sigma}_n\}_{n=1}^{N^2}$ of the set of Hermitian $N \times N$ operators to get the result.

Secondly, we consider the solution to the Liouville-Lindblad equation. As H and V commute, they can be simultaneously diagonalized and there exists a set of projectors $\{\Pi_k(H)\}_{k=1}^{\bar{N}\leq N}$ onto the (orthogonal) simultaneous eigenspaces of H and V such that $\sum_{k=1}^{\bar{N}} \Pi_k(H) = I_{\mathcal{C}^N}$ is a resolution of the identity on the full Hilbert space $\mathbf{H}$ and

$$H = \sum_{k=1}^{\bar{N}} \lambda_k(H)\Pi_k(H), \quad V = \sum_{k=1}^{\bar{N}} \lambda_k(V)\Pi_k(H),$$

where $\lambda_k(H)$ and $\lambda_k(V)$ are the respective real eigenvalues of H and V, and $\bar{N} \leq N$ is the number of *distinct* eigenvalues of H.

Pre-/post-multiplying the master Eq. (3) with Lindblad term (2) by $\Pi_k(H)$ and $\Pi_\ell(H)$, respectively, yields

$$\Pi_k(H)\dot{\rho}(t)\Pi_\ell(H) = (-\imath\omega_{k\ell} + \delta\gamma_{k\ell})\Pi_k(H)\rho(t)\Pi_\ell(H), \tag{21}$$

with $\omega_{k\ell} = \lambda_k(H) - \lambda_\ell(H)$ and $\gamma_{k\ell} = -\frac{1}{2}(\lambda_k(V) - \lambda_\ell(V))^2 \leq 0$. The solution to this equation is

$$\Pi_k(H)\rho(t)\Pi_\ell(H) = e^{-t(\imath\omega_{k\ell} - \delta\gamma_{k\ell})}\Pi_k(H)\rho_0\Pi_\ell(H).$$

Since $\sum_{k=1}^{\bar{N}} \Pi_k(H) = I$, the full solution is found as $\rho(t) = \sum_{k,\ell=1}^{\bar{N}} \Pi_k(H)\rho(t)\Pi_\ell(H)$, which yields

$$\rho(t) = \sum_{k,\ell=1}^{\bar{N}} e^{-t(\imath\omega_{k\ell} - \delta\gamma_{k\ell})}\Pi_k(H)\rho_0\Pi_\ell(H). \tag{22}$$

Moreover, from the above it is easily verified that

$$\Pi_k \rho(t) \Pi_k = \Pi_k \rho_0 \Pi_k, \quad k = 1, \ldots, \bar{N}.$$

Therefore, remembering that $\sum_{k=1}^{\bar{N}} \Pi_k(H) = I_{\mathcal{C}^N}$, the solution $\rho(t)$ has $\bar{N}$ constant directions.

Theorem 3: Let $\bar{N} = N$. If $[H, V] = 0$ in the quantum master Equation (3), then $[A, S] = 0$ in the Bloch equation and furthermore the kernels of A and S coincide and are both N-dimensional.

For A and S to commute, we have to show that they have the same eigenspaces. Let us begin with the kernel, the eigenspace associated with the 0-eigenvalue. Recall that the invariant directions are $\{\Pi_k \rho_0 \Pi_k\}_{k=1}^{N}$. Therefore, in the Bloch representation, we easily find a basis for the kernel of $A + \delta S$:

$$\boldsymbol{u}_{N^2-N+k} = (\mathrm{Tr}((\Pi_k \rho_0 \Pi_k)\boldsymbol{\sigma}_n))_{n=1}^{N^2}, \quad k = 1, \ldots, N.$$

Note that this kernel basis does not depend on δ and therefore this is the common kernel of A and S. We further have the freedom to orthonormalize this basis using the Gram-Schmidt process. Next, regarding the generically nonvanishing eigenvalues, elementary manipulations show that

$$(-\imath \,\mathrm{Ad}_H + \delta \mathfrak{L}(V))(\Pi_k \rho_0 \Pi_\ell) = (-\imath \omega_{k\ell} + \delta \gamma_{k\ell})(\Pi_k \rho_0 \Pi_\ell).$$

In other words, $(\Pi_k \rho_0 \Pi_\ell)$ is an eigenvector of the right-hand side of the quantum master Equation (3) associated with the eigenvalue $-\imath \omega_{k\ell} + \delta \gamma_{k\ell} \neq 0$. In the Bloch representation, therefore, the eigenvectors associated with the nonvanishing eigenvalues of $A + \delta S$ are

$$\boldsymbol{u}_{k \neq \ell} = (\mathrm{Tr}((\Pi_k \rho_0 \Pi_\ell)\boldsymbol{\sigma}_n))_{n=1}^{N^2}, \quad 1 \leq k \neq \ell \leq N. \tag{23}$$

The eigenvalues remain the same, as can be seen from the commutativity of the following diagram:

$$\begin{array}{ccc} \mathbf{Herm} & \overset{-\imath \,\mathrm{Ad}_H + \delta \mathfrak{L}(V)}{\longrightarrow} & \mathbf{Herm} \\ \downarrow & & \downarrow \\ \mathbb{R}^{N^2} & \overset{A+\delta S}{\longrightarrow} & \mathbb{R}^{N^2} \end{array}$$

together with the linearity of the Bloch representation $\downarrow$. Let us relabel the eigenvectors in Eq. (23) as $\{\boldsymbol{u}_n\}_{n=1}^{N^2-N}$. By Lemma 2, this set is orthonormal. Together with the kernel, they define a unitary matrix

$$U = \begin{pmatrix} \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_{N^2-N} & \boldsymbol{u}_{N^2-N+1} & \cdots & \boldsymbol{u}_{N^2} \end{pmatrix}$$

that diagonalizes $A + \delta S$ for all δ. Again, it is important to observe that U does not depend on δ. It remains to show that such transformation *simultaneously* diagonalizes A and S. Setting

$\delta = 0$ implies that U diagonalizes A. Then note that U also diagonalizes $\frac{1}{\delta+1}A + \frac{\delta}{1+\delta}S$. Setting $\delta \uparrow \infty$ implies that U diagonalizes S as well. Therefore A and S have the same eigenvectors and hence $[A, S] = 0$.

Remark 3: Lemma 1 in its second form is known, although proved via minimal polynomial methods [16] rather that via invariant subspaces, as done in the Appendix. The statement of the lemma in its first form is, however, believed to be novel.

B. Pure Dephasing as Perturbation of Hamiltonian Dynamics

The above formulation allows us to study the performance of a quantum process under pure dephasing. In this case A is the Bloch matrix for a system subject to Hamiltonian dynamics and the perturbation S is the Bloch representation of the pure dephasing $\mathfrak{L}(V)$ with strength δ. Recalling $[H, V] = 0$, then by Th. 3, A and S are $N^2 \times N^2$ matrices of rank $\leq N^2 - N$ with equality in the generic case. More specifically, A and S are simultaneously diagonalizable [5] by a complex unitary matrix U,

$$U^\dagger A U = \operatorname{diag}(\Omega, 0), \tag{24a}$$

$$U^\dagger S U = \operatorname{diag}(\Gamma, 0), \tag{24b}$$

where Ω and Γ are diagonal matrices of rank $N^2 - N$ in the generic case, with purely imaginary entries $\imath\omega_{k\neq\ell} = \imath(\lambda_k(H) - \lambda_\ell(H))$ for Ω and purely real and negative entries $\gamma_{k\neq\ell} = -\frac{1}{2}(\lambda_k(V) - \lambda_\ell(V))^2$ for Γ (see [5]). This allows us to rewrite Eqs. (10a) and (10b) as

$$U^\dagger \dot{\boldsymbol{z}} = U^\dagger (A + \delta S)\boldsymbol{z} + \delta U^\dagger S \boldsymbol{w}_u(t), \tag{25a}$$

$$U^\dagger \dot{\boldsymbol{z}} = U^\dagger A \boldsymbol{z} + \delta U^\dagger S \boldsymbol{w}_p(t). \tag{25b}$$

By setting

$$\boldsymbol{\zeta} = U^\dagger \boldsymbol{z}, \quad \boldsymbol{v}_u = U^\dagger \boldsymbol{w}_u, \quad \boldsymbol{v}_p = U^\dagger \boldsymbol{w}_p, \tag{26}$$

we obtain

$$\dot{\boldsymbol{\zeta}} = \operatorname{diag}(\Omega + \delta\Gamma, 0)\boldsymbol{\zeta} + \delta \operatorname{diag}(\Gamma, 0)\boldsymbol{v}_u, \tag{27a}$$

$$\dot{\boldsymbol{\zeta}} = \operatorname{diag}(\Omega, 0)\boldsymbol{\zeta} + \delta \operatorname{diag}(\Gamma, 0)\boldsymbol{v}_p. \tag{27b}$$

Note that, despite the *real* form of the Bloch equations, $\boldsymbol{\zeta}$ and $\boldsymbol{v}$ are complex, as $U^\dagger$ is in general a complex unitary operator, although we could easily define an equivalent real form. Finally, we can partition the vectors $\boldsymbol{\zeta}$ and $\boldsymbol{v}$ so that

$$\begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\zeta}_1 \\ \boldsymbol{\zeta}_2 \end{pmatrix} = \begin{pmatrix} \Omega\boldsymbol{\zeta}_1 \\ 0 \end{pmatrix}, \tag{28a}$$

$$\begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix} = \begin{pmatrix} \Gamma\boldsymbol{v}_1 \\ 0 \end{pmatrix}. \tag{28b}$$

We clearly have $\dot{\boldsymbol{\zeta}}_2 = 0$, i.e., $\boldsymbol{\zeta}_2(t)$ is constant. Therefore the dynamics of the system are completely determined by $\boldsymbol{\zeta}_1(0)$ and the reduced model Bloch equation

$$\dot{\boldsymbol{\zeta}}_1 = (\Omega + \delta\Gamma)\boldsymbol{\zeta}_1 + \delta\Gamma\boldsymbol{v}_{u,1}, \tag{29a}$$

$$\dot{\boldsymbol{\zeta}}_1 = \Omega\boldsymbol{\zeta}_1 + \delta\Gamma\boldsymbol{v}_{p,1}. \tag{29b}$$

Generally both Ω and $\Omega + \delta\Gamma$ are invertible, and taking the Laplace transform yields

$$\widehat{\boldsymbol{\zeta}}_1(s) = (sI - \Omega - \delta\Gamma)^{-1}\delta\Gamma\hat{\boldsymbol{v}}_{u,1}(s), \tag{30a}$$

$$\widehat{\boldsymbol{\zeta}}_1(s) = (sI - \Omega)^{-1}\delta\Gamma\hat{\boldsymbol{v}}_{p,1}(s). \tag{30b}$$

Therefore,

$$\widehat{\boldsymbol{\zeta}}_1 = T^u_{\boldsymbol{\zeta}_1,\boldsymbol{v}_{u,1}}(s)\hat{\boldsymbol{v}}_{u,1} = T^p_{\boldsymbol{\zeta}_1,\boldsymbol{v}_{p,1}}(s)\hat{\boldsymbol{v}}_{p,1}, \tag{31}$$

where the transfer function from the input $\hat{\boldsymbol{v}}_{u,1}(s)$ or $\hat{\boldsymbol{v}}_{p,1}(s)$ to the state $\hat{\boldsymbol{\zeta}}_1(s)$ is

$$T^u_{\boldsymbol{\zeta}_1,\boldsymbol{v}_{u,1}}(s) = (sI - \Omega - \delta\Gamma)^{-1}\delta\Gamma, \tag{32a}$$

$$T^p_{\boldsymbol{\zeta}_1,\boldsymbol{v}_{p,1}}(s) = (sI - \Omega)^{-1}\delta\Gamma. \tag{32b}$$

Taking $\Omega = \operatorname{diag}(\imath\omega_{k,\ell})$ and $\Gamma = \operatorname{diag}(\gamma_{k,\ell}) < 0$ we obtain

$$\begin{aligned} T^u_{\boldsymbol{\zeta}_1,\boldsymbol{v}_{u,1}}(\imath\omega,\delta) &= \operatorname{diag}((\imath\omega - \imath\omega_{k\ell} - \delta\gamma_{k\ell})^{-1})\operatorname{diag}(\delta\gamma_{k\ell}) \\ &= \operatorname{diag}\left(\frac{\delta\gamma_{k\ell}}{\imath\omega - \imath\omega_{k\ell} - \delta\gamma_{k\ell}}\right). \end{aligned}$$

Taking the norm to be the largest singular value yields

$$\|T^u_{\boldsymbol{\zeta}_1,\boldsymbol{v}_{u,1}}(\imath\omega,\delta)\|_\infty = \max_{\omega,(k,\ell)}\left|\frac{\delta\gamma_{k\ell}}{\imath(\omega - \omega_{k\ell}) - \delta\gamma_{k\ell}}\right| = 1, \tag{33}$$

where the bound is obtained for $\omega = \omega_{k,\ell}$, i.e., if ω is an eigenfrequency of the system, for all δ, including the limit $\delta \to 0$.

The problem can also be tackled using the robust performance approach of Sec. III-A culminating in Th. 2. Setting $\Phi(s) = sI - \Omega$, we have

$$G = \begin{pmatrix} \Phi^{-1}(s)\Gamma & \Phi^{-1}(s)\Gamma \\ I & 0 \end{pmatrix}, \quad \boldsymbol{\Delta} = \operatorname{diag}(\delta I, \Delta_f),$$

and

$$\det(I + G\boldsymbol{\Delta}) = \det[I + \delta\Phi^{-1}(s)\Gamma - \delta\Phi^{-1}(s)\Gamma\Delta_f].$$

If the fictitious feedback Δ_f is diagonal, it can be shown (although not completely trivially, as shown in the Appendix) that the optimal Δ_f yields $\|T^u_{\boldsymbol{\zeta}_1,\boldsymbol{v}_{u,1}}\| = 1/\|\Delta_f\|$. With this diagonal structure, the above determinant vanishes if

$$1 + \delta\frac{\gamma_{k\ell}}{s - \imath\omega_{k\ell}} - \delta\frac{\gamma_{k\ell}}{s - \imath\omega_{k\ell}}(\Delta_f)_{k\ell} = 0,$$

for some k, ℓ. The above can be solved for $(\Delta_f)_{k\ell}$ as

$$(\Delta_f)_{k\ell} = \frac{s - \imath\omega_{k\ell}}{\delta\gamma_{k\ell}} + 1,$$

which assumes its minimum of 1 for $s = \imath\omega_{k\ell}$ and $\forall\delta > 0$. Choosing $\delta \leq 1$ yields $\min\|\boldsymbol{\Delta}\| = 1$.

Thus, $\mu_{\mathcal{D}}(G(s)) = 1$, which is consistent with Eq. (33). Note that relaxation of the diagonal structure of Δ_f to a fully populated matrix would yield $\|T^u_{\boldsymbol{\zeta}_1,\boldsymbol{v}_{u,1}}\|$, as is well known, and this would not change the result.

The calculation for $T^p_{\boldsymbol{\zeta}_1,\boldsymbol{v}_{p,1}}$ is similar, but without the γ terms in the denominator, and the norm diverges when the frequencies are resonant.

C. Hamiltonian Uncertainty and Dephasing as Perturbations

If there is uncertainty in the Hamiltonian in addition to dephasing, the situation becomes more complex. Here, we model Hamiltonian uncertainty by parameterizing the Hamiltonian. That is, assume H_ε depends continuously on $\varepsilon \in \mathcal{E}$ where $\mathcal{E}$ is compact and comprises 0 in its interior where H_0 is the nominal Hamiltonian. To enforce dephasing acting in the eigenbasis of the Hamiltonian, we must choose a quantum jump operator V_ε satisfying $[H_\varepsilon, V_\varepsilon] = 0$ that depends continuously on ε. If H_ε is continuous, under the condition that $\dim\varepsilon \leq 3$, generically, the eigenvalues of H_ε do not cross [10], [11], [17]. Therefore, the eigenvectors $\{\boldsymbol{u}_k(\varepsilon)\}_{k=1}^N$ of H_ε form a continuous orthonormal basis and

$$V_\varepsilon = U_\varepsilon^\dagger \operatorname{diag}(\lambda_1(V_\varepsilon), \cdots, \lambda_N(V_\varepsilon))U_\varepsilon,$$

where

$$U_\varepsilon = \Big(\boldsymbol{u}_1(\varepsilon) \quad \cdots \quad \boldsymbol{u}_N(\varepsilon)\Big)$$

and the eigenvalues $\lambda_k(V_\varepsilon)$ can be chosen arbitrarily provided they are real, positive and remain simple as ε varies (that is, the eigenvalues $\lambda_k(V_\varepsilon)$ do not cross under varying ε).

Going to the Bloch representation, we define $\omega_{k\ell}(\varepsilon) = \lambda_k(H_\varepsilon) - \lambda_\ell(H_\varepsilon)$ and $\gamma_{k\ell}(\epsilon) = -\frac{1}{2}(\lambda_k(V_\varepsilon) - \lambda_\ell(V_\varepsilon))^2$. If A_ε and S_ε are the Bloch representations of H_ε and $\mathfrak{L}(V_\varepsilon)$, resp., by the proof of Theorem 3, the $k \neq \ell$ eigenvalues $-(\imath\omega_{k\ell}(\varepsilon) + \delta\gamma_{k\ell}(\varepsilon))$ of $A_\varepsilon + \delta S_\varepsilon$ do not cross. Therefore, we can simultaneously diagonalize A_ε and S_ε, hence $A_\varepsilon + \delta S_\varepsilon$ for all ε with the unitary operators U_ε. The latter operator happens to be continuous under the no-crossing hypothesis,

$$U_\varepsilon^\dagger(A_\varepsilon + \delta S_\varepsilon)U_\varepsilon = \mathrm{diag}(\Omega_\varepsilon + \delta\Gamma_\varepsilon, 0), \tag{34}$$

where Ω_ε, Γ_ε display the perturbed eigenfrequencies, dampings, resp., on their diagonals. We can proceed as before and set

$$\boldsymbol{\zeta}_\varepsilon = U_\varepsilon^\dagger \boldsymbol{z}_\varepsilon, \quad \boldsymbol{\upsilon}_\varepsilon = U_\varepsilon^\dagger \boldsymbol{w}, \tag{35}$$

to obtain, in the unperturbed case,

$$\dot{\boldsymbol{\zeta}}_\varepsilon = U_\varepsilon^\dagger(A_\varepsilon + \delta S_\varepsilon)U_\varepsilon U_\varepsilon^\dagger \boldsymbol{z} + \delta U_\varepsilon^\dagger S U_\varepsilon U_\varepsilon^\dagger \boldsymbol{w}_u, \tag{36}$$

$$= \mathrm{diag}(\Omega_\varepsilon + \delta\Gamma_\varepsilon, 0)\boldsymbol{\zeta}_\varepsilon + \mathrm{diag}(\delta\Gamma_\varepsilon, 0)\boldsymbol{\upsilon}_\varepsilon, \tag{37}$$

where $\boldsymbol{\zeta}_\varepsilon$ and $\boldsymbol{\upsilon}_\varepsilon$ are complex as $U_\varepsilon^\dagger$ is a complex unitary operator.

Note that should crossing of eigenvalues occur, we can still proceed with block-diagonalization invoking Doležal's theorem [18].

V. General Dissipative Dynamics

We extend and apply the formalism of Sec. III to the general case of dissipative systems where $[H, V] \neq 0$ and where the uncertainty manifests as decoherence and/or in the Hamiltonian. We therefore slightly generalize the uncertainty structure developed in Sec. II, Eq. (5). As in Sec. IV-C, the Hamiltonian H_p depends continuously on a parameter p and, in addition to a global decoherence rate δ, the jump operator V_q will be made dependent on a parameter q to allow for collective versus independent dissipation. This results in the $(A + \delta\mathcal{S})$ Bloch matrix having generic rank $N^2 - 1$ because of the constancy of the trace, where the nominal A comprises *both* the nominal Hamiltonian and decoherence dynamics, and all uncertainties are relegated to $\delta\mathcal{S}$.

The main objective of this section is to address difficulties in setting up robust performance due to the rank deficiency of the Bloch matrix. Similar difficulties in classical systems are reported in [3] and circumvented by decoupling the design at low frequency from the design at medium to high frequency. Here we address such issues *exactly* at $s = 0$.

Despite its inconvenience, the rank deficiency of the $N^2 \times N^2$ matrix $A+\delta\mathcal{S}$ can be exploited. It is customary in physics to define a reduced $(N^2-1)\times(N^2-1)$ Bloch matrix $\overline{A+\delta\mathcal{S}}$ of full rank. This leads to an inhomogeneous Bloch equation for the reduced Bloch vector $\boldsymbol{s}$ where the trace component of $\boldsymbol{r}$ in Eq. (4) has been removed:

$$\tfrac{d}{dt}\boldsymbol{s}(t) = (\overline{\mathcal{A}+\delta\mathcal{S}})\boldsymbol{s}(t) + \boldsymbol{c}. \tag{38}$$

This equation is useful in some regards. If $\overline{A+\delta\mathcal{S}}$ is invertible (generic case) then the system has a unique steady state $\boldsymbol{s}_{\rm ss} = -(\overline{A+\delta\mathcal{S}})^{-1}\boldsymbol{c}$ and it can be shown that this state is *globally* asymptotically stable [19]. Therefore, the steady state is independent of the initial state and completely robust to initial state preparation errors. Indeed, a key application of this scheme is quantum state preparation. However, the scheme is sensitive to uncertainty in both, the Hamiltonian and the dissipative processes.

A. Noise Transmission

To deal with the singularity at $s = 0$ we define a "pseudo-inverse" that leads to a matrix "pseudo-inversion" lemma.

Consider Eq. (11) in the generalized uncertainty structure with no initial state preparation error. The problem of the singularity of $\Phi(s) := sI - A$ at $s = 0$ raised in Sec. III-A can be traced back to whether the equation

$$(\Phi(s) - \delta\mathcal{S})\hat{\boldsymbol{z}}(s) = \delta\mathcal{S}\hat{\boldsymbol{w}}_{\boldsymbol{u}}(s) \tag{39}$$

is solvable for $\hat{\boldsymbol{z}}(s)$ when the invertibility of $\Phi(s) - \delta\mathcal{S}$ is not guaranteed.

One might wish to utilize existing matrix pseudo-inversion lemmas such as [1], [2]: $(A + \delta\mathcal{S})^\ddagger = A^\ddagger - A^\ddagger\delta\left(I + \mathcal{S}A^\ddagger\delta\right)^\ddagger\mathcal{S}A^\ddagger$ where $(\cdot)^\ddagger$ is the Moore-Penrose pseudo-inverse. Unfortunately, this lemma in general fails in the present situation, as it requires that A and $\mathcal{S}$ are Hermitian, nonnegatively defined and under a condition reminiscent of $\mathrm{Col}\{\mathcal{S}\} \subseteq \mathrm{Col}\{A\}$. However, we can define a suitable #-inverse by considering the structure of the problem. Specifically, we have

Lemma 3: For the Bloch representation $A+\delta\mathcal{S}$ relative to the Pauli or Gell-Mann basis where $\sigma_{N^2}=\frac{1}{\sqrt{N}}I$, which implies that $\boldsymbol{r}_{N^2}=\mathrm{Tr}(\rho)\equiv 1$, the last rows of A and $\mathcal{S}$ vanish. In the case of dephasing in the Hamiltonian eigenbasis, considered previously, the last column vanishes as well. In general, the last column involves the decoherence rates.

From the above lemma, it follows that the matrices $\Phi(0)$ and $\mathcal{S}$ are structured as follows:

$$-A=\Phi(0)=\left(\begin{array}{c|c}\Phi_{11} & \phi_{12}\\ \hline 0 & 0\end{array}\right),\quad \mathcal{S}=\left(\begin{array}{c|c}\mathcal{S}_{11} & \mathcal{S}_{12}\\ \hline 0 & 0\end{array}\right). \tag{40}$$

Hence, under the assumption that $\Phi_{11}(0)-\delta\mathcal{S}_{11}$ is invertible, Eq. (39) has a (nonunique) solution. Here we retain the minimum norm solution $\hat{\boldsymbol{z}}(s)=\begin{pmatrix}\hat{\boldsymbol{z}}_1(s)^T & 0^T\end{pmatrix}^T$ for the reason that together with a matrix pseudo-inversion-like lemma it leads to a robust performance result. The key is to define an operator $(\cdot)^{\#}$ such that the minimum norm solution can be expressed as

$$\hat{z}(s)=(\Phi(0)-\delta\mathcal{S})^{\#}\delta\mathcal{S}\hat{\boldsymbol{u}}(s)$$

Definition 2: For a general matrix structured as

$$M=\left(\begin{array}{c|c}M_{11} & m_{12}\\ \hline 0 & 0\end{array}\right),$$

where M_{11} is invertible, the #-inverse is defined as

$$M^{\#}=\left(\begin{array}{c|c}M_{11}^{-1} & 0\\ \hline 0 & 0\end{array}\right).$$

Note that this is *not* the Moore-Penrose pseudo-inverse.

To summarize, we have

Theorem 4: With matrices $\Phi(0)$ and $\mathcal{S}$ structured as Eq. (40), under the assumption that $\Phi_{11}(0)-\delta\mathcal{S}_{11}$ is invertible, the minimum norm solution to Eq. (39) can be written as $\hat{z}(s)=(\Phi(0)-\delta\mathcal{S})^{\#}\delta\mathcal{S}\hat{\boldsymbol{w}}(s)$.

With this result, we *define*

$$T_{\boldsymbol{z},\boldsymbol{w}_u}(0)=(\Phi(0)-\delta\mathcal{S})^{\#}\delta\mathcal{S}.$$

To get to robust performance, observe that the #-inversion has a matrix #-inversion lemma,

$$T_{\boldsymbol{z},\boldsymbol{w}_u}(0)=(\Phi(0)-\delta\mathcal{S})^{\#}\delta\mathcal{S}=(I-\Phi(0)^{\#}\delta\mathcal{S})^{-1}\Phi(0)^{\#}\delta\mathcal{S},$$

and compare it with Eq. (13). Following the same path, instead of Eq. (14), we define

$$G_{\boldsymbol{z},\boldsymbol{w}_u}(0) = \begin{pmatrix} \Phi(0)^{\#}\mathcal{S} & \Phi(0)^{\#}\mathcal{S} \\ I & 0 \end{pmatrix}.$$

The robust performance theorem at $s = 0$ is a straightforward adaptation of Th. 2 and we have

Corollary 4: $\lim_{s\to 0} G_{\boldsymbol{z},\boldsymbol{w}_u}(s) = G_{\boldsymbol{z},\boldsymbol{w}_u}(0)$.

In [6], this result was approached heuristically by $s \downarrow 0$.

B. Initial State Preparation Error Response

Finally, considering the initial state preparation error response, the coefficient matrices $\Phi(0) = -A$ and $A+\delta\mathcal{S}$ in Eq. (17) are singular with rank N^2-1, δ-generically, as in the previous case. However, we can again solve Eq. (17) in its enhanced perturbation structure, by remembering that the last (N^2th) row of A and $\mathcal{S}$ vanish as a corollary of the constancy of the trace, and that $z_{N^2}(0) = 0$ as any prepared state must be represented by a density of trace 1. Therefore, Eq. (17) has the exact solution $\hat{\boldsymbol{z}}(0) = (-A-\delta\mathcal{S})^{\#}\boldsymbol{z}(0)$, leading to the transfer matrix $T^u_{\boldsymbol{z},\boldsymbol{z}_0}(0) = (-A-\delta\mathcal{S})^{\#}$.

Lemma 4: Under the same assumptions as in Th. 4 and provided $I + \mathcal{S}A^{\#}\delta$ is invertible, as it generically is, the matrix $\#$-inversion lemma

$$(A+\delta\mathcal{S})^{\#} = A^{\#} - A^{\#}\delta\left(I + \mathcal{S}A^{\#}\delta\right)^{-1}\mathcal{S}A^{\#}$$

holds.

It follows from the lemma that the uncertainty can be represented as diagonal feedback $\eta = (\delta I)v$ wrapped around

$$\begin{pmatrix} v \\ \hat{\boldsymbol{z}}(s) \end{pmatrix} = \underbrace{\begin{pmatrix} \mathcal{S}\Phi(0)^{\#} & \mathcal{S}\Phi(0)^{\#} \\ \Phi(0)^{\#} & \Phi(0)^{\#} \end{pmatrix}}_{G^0_{\boldsymbol{z},\boldsymbol{z}_0}} \begin{pmatrix} \eta \\ \boldsymbol{z}_0 \end{pmatrix},$$

as the substitute for Eq. (19). The chief difference relative to the $G_{\boldsymbol{z},\boldsymbol{w}_u}$ case is that, contrary to Corollary 4, there are cases of discontinuity, viz., $\lim_{s\to 0} G_{\boldsymbol{z},\boldsymbol{z}_0}(s) \neq G^0_{\boldsymbol{z},\boldsymbol{z}_0}$, as the cavity example in the next section exposes.

VI. Application: Two qubits in a cavity

We apply the approach to a simple quantum system of two two-level atoms in a lossy cavity designed to maximize entanglement generation between the atoms, or more broadly in the quantum Internet [20]. The entanglement between the two qubits is measured by the *concurrence*

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{41}$$

where ρ is the density operator of the two qubits and the λ_k's are the eigenvalues in decreasing order of $\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ [21], where $\rho^* = \mathrm{conj}(\rho)$, and

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & \imath \\ -\imath & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{42}$$

are the Pauli operators. Observe the slightly unusual way to write them.

The dynamics are described by the Lindblad equation

$$\dot{\rho}(t) = -\imath[H_{\alpha,\Delta}, \rho(t)] + \mathfrak{L}(V_\gamma)\rho(t). \tag{43}$$

$H_{\alpha,\Delta}$ is the Hamiltonian

$$H_{\alpha,\Delta} = \sum_{n=1}^{2} \left(\alpha_n^* \sigma_+^{(n)} + \alpha_n \sigma_-^{(n)} + \Delta_n \sigma_+^{(n)} \sigma_-^{(n)} \right),$$

where $\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is the raising operator, $\sigma_- := \sigma_+^\dagger$ is the lowering operator, $\sigma_\pm^{(1)} = \sigma_\pm \otimes I_{2\times 2}$, and $\sigma_\pm^{(2)} = I_{2\times 2} \otimes \sigma_\pm$. The super-operator $\mathfrak{L}(V)\rho = V\rho V^\dagger - \frac{1}{2}(V^\dagger V\rho + \rho V^\dagger V)$ is a classical Lindbladian with dissipation $V_\gamma = \sum_{n=1}^{2} \gamma_n \sigma_-^{(n)}$. After some calculations, it is found that

$$H_{\alpha,\Delta} = \begin{pmatrix} 0 & \alpha_2 & \alpha_1 & 0 \\ \alpha_2^* & \Delta_2 & 0 & \alpha_1 \\ \alpha_1^* & 0 & \Delta_1 & \alpha_2 \\ 0 & \alpha_1^* & \alpha_2^* & \Delta_1 + \Delta_2 \end{pmatrix}, \tag{44}$$

$$V_\gamma = \begin{pmatrix} 0 & \gamma_2 & \gamma_1 & 0 \\ 0 & 0 & 0 & \gamma_1 \\ 0 & 0 & 0 & \gamma_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{45}$$

Note that $[H_{\alpha,\Delta}, V_\gamma] \neq 0$. For anti-symmetric detuning, $\Delta_1 = -\Delta_2 = \Delta$, and symmetric driving, $\alpha_1 = \alpha_2 = \alpha$, $\gamma_1 = \gamma_2 = \gamma$ and Δ, α and γ real, it is easy to show that

$$|\Psi_{\rm ss}\rangle = \frac{1}{\sqrt{\Delta^2 + 2\alpha^2}} \begin{pmatrix} \Delta \\ \alpha \\ -\alpha \\ 0 \end{pmatrix}$$

is a steady state of the system, as $H_{\alpha,\Delta} |\Psi_{\rm ss}\rangle = \mathbf{0}$ and $V_\gamma |\Psi_{\rm ss}\rangle = \mathbf{0}$ and thus $\rho_{\rm ss} = |\Psi_{\rm ss}\rangle \langle\Psi_{\rm ss}|$ satisfies $\dot{\rho}_{\rm ss} = 0$.

Since (as we shall explain below) this steady state is generically globally attractive, any initial state converges to it, and its concurrence determines the performance. It can be shown that the concurrence of the steady state is [22]

$$C_{\rm ss} := C(\rho_{\rm ss}) = \frac{1}{\frac{1}{2}(\Delta/\alpha)^2 + 1}.$$

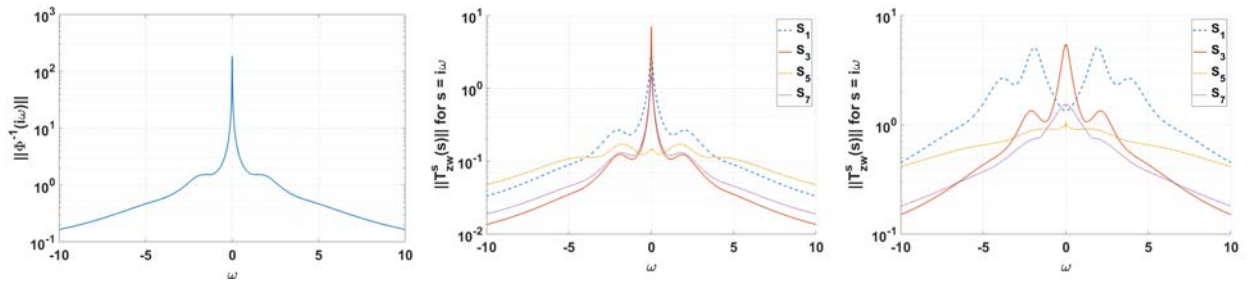
To maximize the concurrence we therefore want Δ/α to be as small as possible. However, in the limit of no detuning, $\Delta \to 0$, the attractivity of the steady state is lost and there are trade-offs in terms of the speed of convergence and robustness.

To examine the system's robustness, we consider the nominal system dynamics in the Bloch formulation,

$$\tfrac{d}{dt}\boldsymbol{r}(t) = A_{\alpha,\Delta,\gamma}\boldsymbol{r}(t), \tag{46}$$

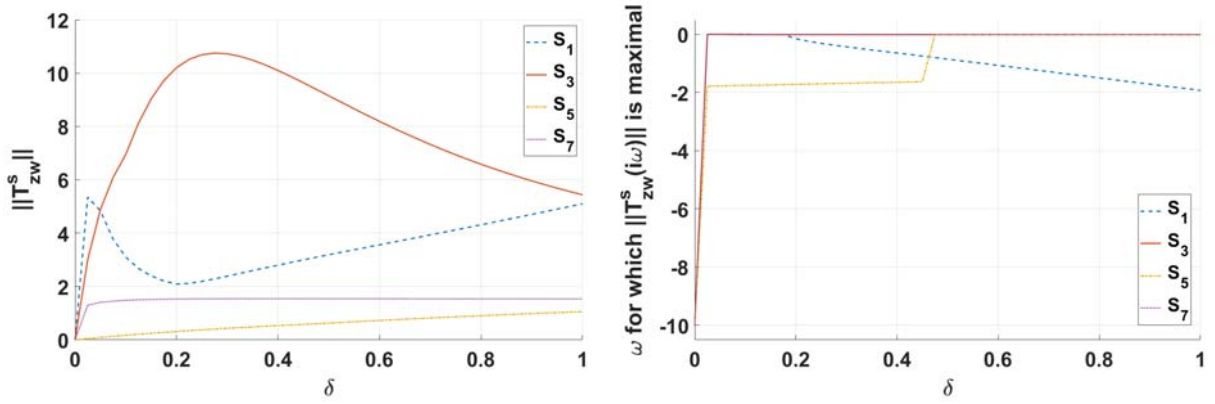
relative to the Pauli basis $\{e_k \otimes e_\ell : k, \ell = 1, \ldots, 4\}$, where $(e_1, e_2, e_3, e_4) = \frac{1}{\sqrt{2}}(I_{2\times 2}, \sigma_x, \sigma_y, \sigma_z)$. It is tacitly assumed that the components are labeled so that $\boldsymbol{z}_{16}$ is the error on $\mathrm{Tr}(\rho)$ and hence vanishes.

$A_{\alpha,\Delta,\gamma}$ is a real 16×16 matrix whose last row vanishes but contrary to the case of pure dephasing, it can be verified by symbolic computation that its last column depends exclusively on γ and does not vanish for $\gamma > 0$. It can further be verified that for $\alpha \neq 0$, $\gamma \neq 0$ and $\Delta \neq 0$ the rank of $A_{\alpha,\Delta,\gamma}$ is 15; more specifically, the eigenvalues of $A_{\alpha,\Delta,\gamma}$ have negative real parts except for one eigenvalue that is always 0 due to the trace constraint for ρ. Generally, we have stability for all non-zero detunings but for $\Delta = 0$ the rank drops to 14, implying that there is a one-dimensional subspace of steady states. $A_{\alpha,\Delta,0}$ corresponds to unitary evolution and hence the eigenvalues of $A_{\alpha,\Delta,0}$ are purely imaginary. Furthermore, $\mathrm{rank}(A_{\alpha,\Delta,0}) = 10$, which is a nongeneric result due to $\alpha_1 = \alpha_2$ and $\Delta_1 = -\Delta_2$ (generically, its rank is 12). Thus, in this case,



(a) $\Delta = 0$ (b) $\Delta = 0.1$ (c) $\Delta = 1$

Fig. 1: Error gain $\|T^{\mathcal{S}}_{\boldsymbol{z},\boldsymbol{w}_u}(\imath\omega)\|$ as a function of frequency for the structured uncertainties in Eq. (48) and different detunings $\Delta \in \{0, 0.1, 1\}$. Due to the pairwise similarities $\mathcal{S}_1 \sim \mathcal{S}_2$, $\mathcal{S}_3 \sim \mathcal{S}_4$, and $\mathcal{S}_6 \sim \mathcal{S}_7$, $\mathcal{S}_2, \mathcal{S}_4$, and $\mathcal{S}_6$ are not plotted.



(a) Maximum gain $\|T^{\delta\mathcal{S}}_{\boldsymbol{z},\boldsymbol{w}_u}(\imath\omega)\|_\infty$ (b) Frequency for which maximum gain $\|T^{\delta\mathcal{S}}_{\boldsymbol{z},\boldsymbol{w}_u}(\imath\omega)\|$ is achieved

Fig. 2: The maximum gain for the structured uncertainties in Eq. (48) suggest that for small δ ($\delta < 0.1$) the system is most sensitive to perturbations of type $\mathcal{S}_1$ while for larger δ sensitivity to $\mathcal{S}_3$ dominates.

we can interpret the decoherence $A_{0,0,\gamma}$ as a stabilizing controller for the plant $A_{\alpha,\Delta,0}$ and state feedback $\boldsymbol{w}(t) = A_{0,0,\gamma}\boldsymbol{r}(t)$.

A. Structured Uncertainties

To assess the robustness, a structured uncertainty $\delta\mathcal{S}$, not limited to decoherence, is added to the nominal A resulting in the perturbed dynamics

$$\tfrac{d}{dt}\boldsymbol{r}(t) = (A_{\alpha,\Delta,\gamma} + \delta\mathcal{S}(\alpha_1,\alpha_2,\Delta_1,\Delta_2;\gamma_1,\gamma_2))\boldsymbol{r}(t), \tag{47}$$

where $\mathcal{S}(\alpha_1,\alpha_2,\Delta_1,\Delta_2;\gamma_1,\gamma_2)$, with $\alpha_i = 0,1, \Delta_i = 0,\pm 1; \gamma_i = 0,1$ is the structure of the perturbation of A brought about by Hamiltonian parameters (those $\neq 0$) allowed to drift or decoherence rates becoming uncertain. Specifically, we define the following structured perturbations:

$$\mathcal{S}_1 = \mathcal{S}(1,0,0,0;0,0), \tag{48a}$$

$$\mathcal{S}_2 = \mathcal{S}(0,1,0,0;0,0), \tag{48b}$$

$$\mathcal{S}_3 = \mathcal{S}(0,0,1,0;0,0), \tag{48c}$$

$$\mathcal{S}_4 = \mathcal{S}(0,0,0,-1;0,0), \tag{48d}$$

$$\mathcal{S}_5 = \mathcal{S}(0,0,0,0;1,1), \tag{48e}$$

$$\mathcal{S}_6 = \mathcal{S}(0,0,0,0;1,0), \tag{48f}$$

$$\mathcal{S}_7 = \mathcal{S}(0,0,0,0;0,1). \tag{48g}$$

Note that $\mathcal{S}_5$ is a *collective* dissipation, while $\mathcal{S}_6$, $\mathcal{S}_7$ are structured perturbations corresponding to *single* qubit spontaneous emission given by the Bloch representations of the dissipation operators

$$V_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{resp.}$$

B. Frequency Responses given Uncertainties

The resulting error gains $\|T^{\mathcal{S}}_{\boldsymbol{z},\boldsymbol{w}_u}(s)\|$, as functions of frequency $s = \imath\omega$, for the seven perturbations are shown in Fig. 1 for nominal plant and controller parameters of $\alpha_1 = \alpha_2 = 1$, $\gamma_1 = \gamma_2 = 1$ and $\Delta_1(= \Delta) = -\Delta_2 \in \{0, 0.1, 1\}$. Due to symmetry, the effects of $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same, and similarly for $\mathcal{S}_3$ and $\mathcal{S}_4$, and $\mathcal{S}_6$ and $\mathcal{S}_7$, respectively. Hence, it suffices to consider four perturbations. Observe that except for $\mathcal{S}_5$ the bound (33) is violated here as

we are dealing with general dissipation, not the dephasing in the Hamiltonian basis. Our low-frequency focus arises due to the fact that in many quantum systems, $1/f$ noise commonly arises in similar systems, e.g. laser flicker noise in atomic clock systems [23] and magnetic flux noise in superconducting qubits [24].

For $\delta = 0$, the gain plot $\|\Phi^{-1}(s)\|$ versus frequency in Fig. 1 shows that the norm is maximal for $s = 0$ but this is not the case in general. For $\Delta = 0.1$ the gain is maximal at $s = 0$, except for $\mathcal{S}_5$, but for $\Delta = 1$ the maximum over $s = \imath\omega$ of $\|T^{\mathcal{S}}_{\boldsymbol{z},\boldsymbol{w}_u}(s)\|$ is not assumed for $s = 0$. Fig. 2 also confirms this. The maximum of $\|T^{\mathcal{S}}_{\boldsymbol{z},\boldsymbol{w}_u}(\imath\omega)\|$ depends on the perturbation $\mathcal{S}_k$ and strength δ and is not always assumed at $\omega = 0$. The plot suggests that the system is more sensitive to perturbations $\mathcal{S}_3$ in the detuning. Note that other work suggests that robust solutions can be found outside of the regime where $\Delta_1 = -\Delta_2$ [21], but such investigations are outside of the scope of this work.

C. Noise Transmission

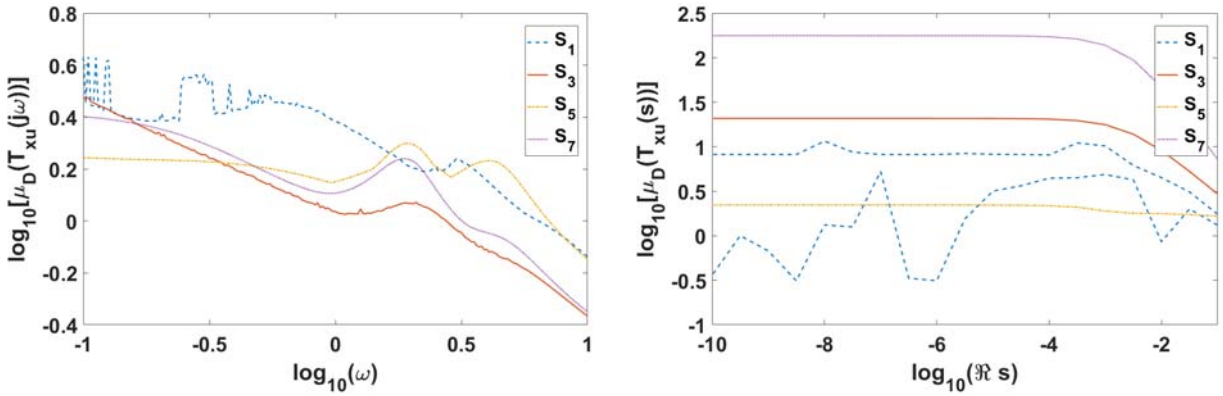
Another way to assess robustness against parameter variation is to assign a bound on the error transmission $\|T_{\boldsymbol{zw}}(s)\|_\infty \leq \mu$, and find out for what δ the error remains below the bound μ. This is essentially what the structured singular value does, as summarized in Th. 2.

Lemma 5: For the cavity example, if $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$, the submatrices Φ_{11} and $\mathcal{S}_{11}$ of $\Phi(0)$ and $\mathcal{S}$, resp., of Eq. (40) are invertible.

Therefore the results of Sec. V-A apply.

Fig. 3a shows simulation results for the structured uncertainties $\mathcal{S}_1, \mathcal{S}_3, \mathcal{S}_5, \mathcal{S}_7$ as a function of frequency on a frequency scale comparable with Fig. 1. Overall, $\mu_{\mathcal{D}}$ behaves well, except for $\mathcal{S}_1$, a fact consistent across this study. Simulation results for the structured singular values as s decreases to 0 along the real axis are shown in Fig. 3b. Except for the lower bound on $\mathcal{S}_1$ they show continuity of $\mu_D(T_{\boldsymbol{z},\boldsymbol{w}_u}(s))$ and the discrepancy between the upper and lower bounds is very mild (not visible on a log-scale), except for $\mathcal{S}_1$. The asymmetry of the perturbation of the driving fields for S_1 is often detrimental to entanglement generation, which may explain its notably different behavior. While this argument is strengthened by the fact that replacing $\mathcal{S}_1$ by a symmetric perturbation of the coupling strengths, $(1, 1, 0, 0; 0, 0)$, makes the behavior disappear, we do not have a full explanation for it.

It is observed that the minimum structured "destabilizing" perturbation $\boldsymbol{\Delta}$ need not be unique for the reason that $G_{\boldsymbol{z},\boldsymbol{w}_u}(0)$ is singular. We argue that this does not cause problems. That



(a) Frequency sweep

(b) $\Re(s)$ sweep

Fig. 3: Bounds on the μ_D of the error transmission $T_{\boldsymbol{z}\boldsymbol{w}_u}$ for the structured uncertainties in Eq. (48) under frequency sweep $s = \imath\omega$ (a, upper bounds only) and for $s \downarrow 0$ along the real axis (b). Upper and lower bounds in (b) coincide for $\mathcal{S}_3$, $\mathcal{S}_5$, and $\mathcal{S}_7$ but diverge for $\mathcal{S}_1$. $\mathcal{S}_1$ displays similarly aberrant behavior under frequency sweep (a).

$G_{\boldsymbol{z},\boldsymbol{w}_u}(0)$ is singular is easily seen from Eq. (40). More specifically, $G_{\boldsymbol{z},\boldsymbol{w}_u}(0)$ has one vanishing row and one vanishing column. This causes the solution to $\det(I + G_{\boldsymbol{z},\boldsymbol{w}_u}(0)\boldsymbol{\Delta}) = 0$ to have a completely arbitrary row and a completely arbitrary column. Part of this nonuniqueness is removed by restricting the solutions to be of minimum norm, but even if the minimum norm solution is not unique, its size $\|\boldsymbol{\Delta}\|$ is uniquely defined.

D. Initial State Preparation Error Response

At $s = 0$, the 16×16 matrices $A_{\alpha,\Delta,\gamma}$ and $A_{\alpha,\Delta,\gamma} + \delta\mathcal{S}_k$ of Eq. (17), evaluated at $(\alpha, \Delta; \gamma) = (1, 1, 0.1, -0.1; 1, 1)$, are singular with rank 15. δ-generically $\mathcal{S}_k$ has rank 15 for $\mathcal{S}_k \neq \mathcal{S}_5$ and rank 14 for $\mathcal{S}_k = \mathcal{S}_5$. The nongeneric δ-values are computed as generalized eigenvalues of $A+\delta\mathcal{S}$ viewed as a pencil of matrices and are given in Table II.

To solve Eq. (17) for the cavity under uncertainty $\mathcal{S}_k$, remember that the last (16^{th}) rows of both A and $\mathcal{S}_k$ vanish as a corollary of the constancy of the trace and that $z_{16}(0) = 0$ provided the prepared state is a genuine density of trace 1. Therefore, Eq. (17) has the exact solution $\hat{\boldsymbol{z}}(0) = (-A - \delta\mathcal{S}_k)^{\#}\boldsymbol{z}(0)$, leading to the transfer matrix $T^u_{\boldsymbol{z},\boldsymbol{z}_0}(0) = (-A - \delta\mathcal{S}_k)^{\#}$. In this case, the matrix $\#$-inversion lemma $(A + \delta\mathcal{S}_k)^{\#} = A^{\#} - A^{\#}\delta\left(I + \mathcal{S}_k A^{\#}\delta\right)^{\#}\mathcal{S}_k A^{\#}$ holds and the

TABLE II: Real, finite generalized eigenvalues of $A + \delta\mathcal{S}_k$.

coefficient matrix	real, finite generalized eigenvalues
$A + \delta\mathcal{S}_1$	none
$A + \delta\mathcal{S}_2$	none
$A + \delta\mathcal{S}_3$	-2, -2
$A + \delta\mathcal{S}_4$	-2, -2
$A + \delta\mathcal{S}_5$	nine-fold 0, ± 1.0989
$A + \delta\mathcal{S}_6$	0.0057, 0.6346, 1.0462, 2.6465
$A + \delta\mathcal{S}_7$	0.0057, 0.6346, 1.0462, 2.6465

uncertainty can be represented as a diagonal feedback $\eta = (\delta I)v$ via

$$\begin{pmatrix} v \\ \hat{\boldsymbol{z}}(s) \end{pmatrix} = \underbrace{\begin{pmatrix} \mathcal{S}_k\Phi(0)^{\#} & \mathcal{S}_k\Phi(0)^{\#} \\ \Phi^{\#}(0) & \Phi^{\#}(0) \end{pmatrix}}_{G^0_{\boldsymbol{z},\boldsymbol{z}_0}} \begin{pmatrix} \eta \\ \boldsymbol{z}_0 \end{pmatrix},$$

as the substitute for Eq. (19). The chief difference relative to the $G_{\boldsymbol{z},\boldsymbol{w}_u}$ case is that, contrary to Corollary 4, $\lim_{s\to 0} G_{\boldsymbol{z},\boldsymbol{z}_0}(s) \neq G^0_{\boldsymbol{z},\boldsymbol{z}_0}$. In fact, $G_{\boldsymbol{z},\boldsymbol{z}_0}(s)$ diverges as $s \to 0$, causing $\mu_D(G_{\boldsymbol{z},\boldsymbol{z}_0}(s))$ to diverge as $s \downarrow 0$, as shown by Fig 4.

E. Concurrence versus Log-Sensitivity

If we measure the performance of the control scheme by the concurrence $C_{\rm ss}$ of the steady-state, as shown by Fig. 5, there is concordance between $C_{\rm ss}$ and its log-sensitivity, i.e. they both decrease with increasing detuning. This is contrary to the classical conflict between the sensitivity function S and its log-sensitivity, the complementary sensitivity T, for which $S + T = 1$ holds. The concordance between the log-sensitivity of $C_{\rm ss}$ and $\max_n\{\Re\lambda_n(A)\}$ could also be interpreted as anticlassical because these are two figures of merit that both improve without conflict as the detuning increases. However, the concordance between $C_{\rm ss}$ and $\max_n\{\Re\lambda_n(A)\}$ is classical since it means the higher the performance the lower the stability margin.

VII. Conclusion

We have developed a general robust performance formalism for controlled quantum processes, adapted to coherent and decoherent quantum systems subject to a variety of structured uncertainties. Besides some uncertain parameters that can be treated classically, quantum systems introduce

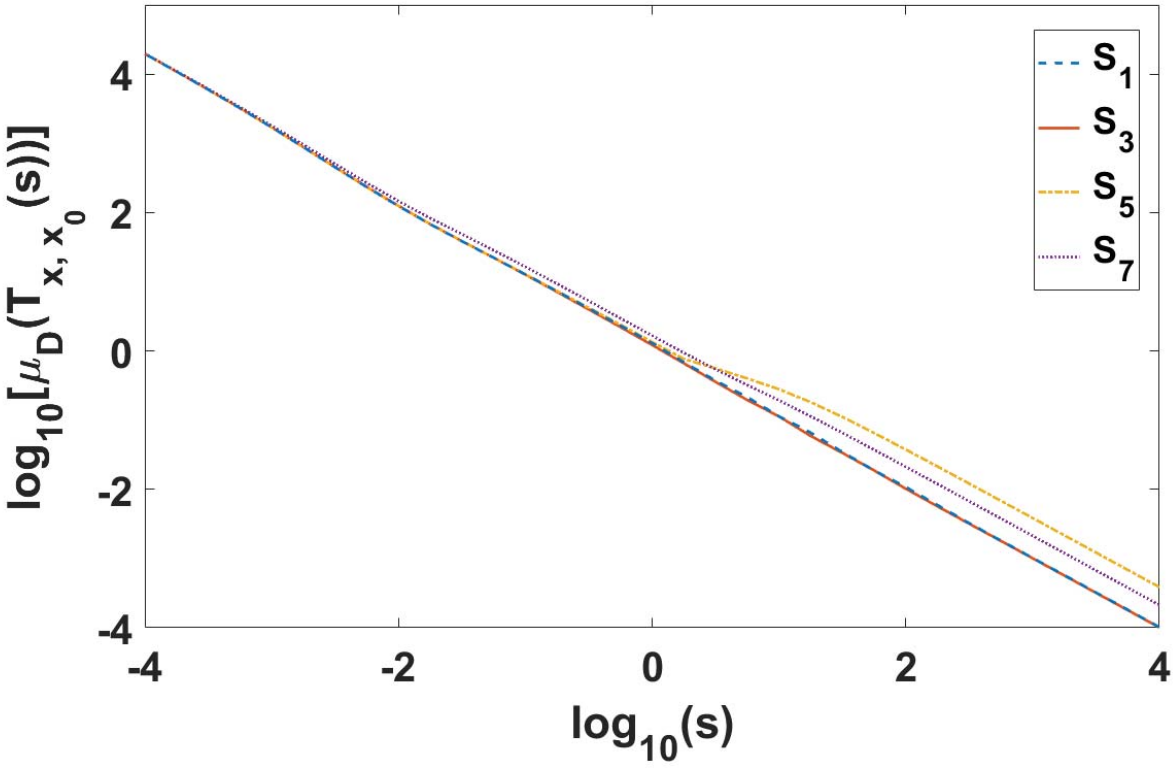


Fig. 4: Upper bounds on μ_D of initial state error transmission $T_{\boldsymbol{z},\boldsymbol{z}_0}(s)$ for the structured uncertainties in Eq. (48) for s decreasing to 0 along the real axis.

uncertainties such as initial state preparation error and decoherence. Another typical feature described by Liouville-like equations is a singularity at $s = 0$ of the open-loop dynamics, due to constancy of the trace. While this poses a challenge for structured singular value analysis, our novel formalism is able to deal with the density operator error response to structured uncertainties, including initial state preparation error.

Proceeding from the general Lindblad equation gives this overall formalism wide physical applicability—to be more specific, XX and Heisenberg chains, quantum spintronic systems, tight-binding models, and cold atoms in optical lattices, to name but a few.

Moreover, we considered the concurrence as the feedback performance in the same way as classical control defines a generalized error as performance. By its very definition, the classical error is linear in the state and the salient difference brought about by the concurrence is that it is nonlinear in the state (and that it has to be maximized rather than minimized). Here, we have

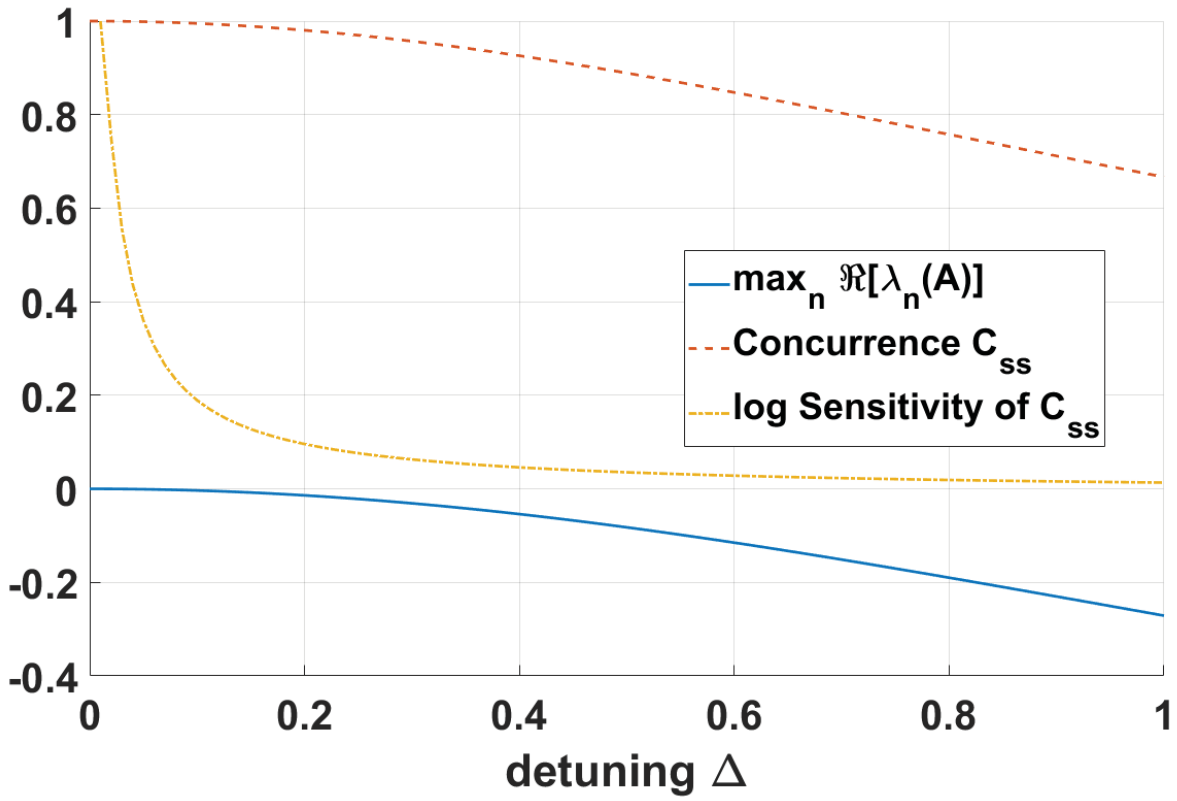


Fig. 5: Maximum real part of eigenvalues of $A+S$, concurrence of steady-state and log sensitivity of steady-state concurrence as a function of detuning for $\alpha = \gamma = 1$. Observe that all three figures of merit are concordant, that is, they all decrease with increasing detuning. Interpreting $C_{\rm ss}$ as performance and $\max_n \Re\lambda_n(A)$ as stability margin, note their anticlassical, nonconflicting behavior.

limited ourselves to add some structured perturbation to the density evolution and observe the resulting variation of concurrence. However, putting a bound on the concurrence and assessing how much structured perturbation would be tolerated before the concurrence bound is reached would be a nonlinear version of the robust performance problem, which we believe is widely open.

APPENDIX

Proof of Lemma 1. Let $P\mathcal{V}_{\lambda_i} = \lambda_i \mathcal{V}_{\lambda_i}$ and $Q\mathcal{W}_{\mu_j} = \mu_j \mathcal{W}_{\mu_j}$ define the eigenvalues λ_i, μ_j and orthonormalized eigenbases $\mathcal{V}_{\lambda_i}, \mathcal{W}_{\mu_j}$ of P and Q, resp. From $PQ = QP$ simple linear algebra

leads to

$$Q\mathcal{V}_{\lambda_i} \subseteq \{\mathcal{V}_{\lambda_i}\}, \quad P\mathcal{W}_{\mu_j} \subseteq \{\mathcal{W}_{\mu_j}\}, \tag{49}$$

where $\{\mathcal{V}_{\lambda_i}\}$, $\{\mathcal{W}_{\mu_j}\}$ denote the *subspaces* spanned by the (orthonormal) columns of $\mathcal{V}_{\lambda_i}$ and $\mathcal{W}_{\mu_j}$, resp.

It follows from Eq. (49), right equation, that $\mathcal{W}_{\mu_j}$ is an invariant subspace of P and therefore it must be made up of eigensubspaces of P. To express $\mathcal{W}_{\mu_j}$ in terms of such eigensubspaces of P, choose a set $I(j)$ of i-indexes such that $\{\mathcal{W}_{\mu_j}\} \subseteq \oplus_{i\in I(j)}\{\mathcal{V}_{\lambda_{I(j)}}\}$. In each $\{\mathcal{V}_{\lambda_{I(j)}}\}$, choose a basis $\bar{\mathcal{V}}_{\lambda_{I(j)}}$ such that

$$\{\mathcal{W}_{\mu_j}\} = \oplus_{i\in I(j)}\{\bar{\mathcal{V}}_{\lambda_{I(j)}}\}. \tag{50}$$

Case #1. Assume both bases $\boxplus_j \mathcal{W}_{\mu_j}$ and $\boxplus_i \mathcal{V}_{\lambda_i}$ are given. In the preceding,

$$\boxplus_i \mathcal{V}_{\lambda_i} = \begin{pmatrix} \mathcal{V}_{\lambda_1} & \mathcal{V}_{\lambda_2} & \cdots \end{pmatrix} \tag{51}$$

is shorthand denoting the various basis elements arranged columnwise in matrix format. To refine the subspace equality to an equality between bases, choose a rotation R_{μ_j} such that

$$\mathcal{W}_{\mu_j} = \left(\boxplus_{i\in I(j)} \bar{\mathcal{V}}_{\lambda_{I(j)}}\right) R_{\mu_j}. \tag{52}$$

Substituting $\mathcal{W}_{\mu_j}$ by its value given by Eq. (52) in the eigenvalue/eigenvector equations yields

$$\begin{aligned} P\left(\boxplus_i \mathcal{V}_{\lambda_i}\right) &= \left(\boxplus_i \mathcal{V}_{\lambda_i}\right) \operatorname{diag}\{\lambda_i\}, \\ Q\left(\boxplus_j \left(\boxplus_{i\in I(j)} \bar{\mathcal{V}}_{\lambda_i} R_{\mu_j}\right)\right) &= \left(\boxplus_j \left(\boxplus_{i\in I(j)} \bar{\mathcal{V}}_{\lambda_i} R_{\mu_j}\right)\right) \operatorname{diag}\{\mu_j\}. \end{aligned}$$

Upon relabeling the first equation, we get

$$\begin{aligned} P\left(\boxplus_j \boxplus_{i\in I(j)} \bar{\mathcal{V}}_{\lambda_i}\right) &= \left(\boxplus_j \oplus_{i\in I(j)} \mathcal{V}_{\lambda_i}\right) \operatorname{diag}\{\Lambda_{I(j)}\}, \\ Q\left(\boxplus_j \left(\boxplus_{i\in I(j)} \bar{\mathcal{V}}_{\lambda_i} R_{\mu_j}\right)\right) &= \left(\boxplus_j \left(\boxplus_{i\in I(j)} \bar{\mathcal{V}}_{\lambda_i} R_{\mu_j}\right)\right) \operatorname{diag}\{\mu_j\}. \end{aligned}$$

Equating the two orthogonal transformation matrices yields

$$\begin{aligned} P\left(\boxplus_j \oplus_{i\in I(j)} \bar{\mathcal{V}}_{\lambda_i}\right) &= \left(\boxplus_j \boxplus_{i\in I(j)} \mathcal{V}_{\lambda_i}\right) \operatorname{diag}\{\Lambda_{I(j)}\}, \\ Q\left(\boxplus_j \left(\oplus_{i\in I(j)} \bar{\mathcal{V}}_{\lambda_i}\right)\right) &= \left(\boxplus_j \left(\boxplus_{i\in I(j)} \bar{\mathcal{V}}_{\lambda_i}\right)\right) \operatorname{diag}\{\mu_j R_{\mu_j}^{\dagger}\}. \end{aligned}$$

In other words, the transformation $\boxplus_j \oplus_{i\in I(j)} \bar{\mathcal{V}}_{\lambda_i}$ simultaneously *block* diagonalizes P and Q.

Case #2. If we utilize the freedom in choosing the basis $\mathcal{V}_{\lambda_i}$, Eq. (52) is simplified to

$$\mathcal{W}_{\mu_j} = \boxplus_{i\in I(j)} \bar{\mathcal{V}}_{\lambda_{I(j)}}.$$

The rotations are no longer needed and P and Q are simultaneously diagonalizable. ■

Proof of Corollary 2. Consider $H_{p^*} = \arg\min_{H_p \in \mathfrak{V}} d(H_p, H_0)$, defining a projection $\pi : \mathcal{H}(N) \to \mathfrak{V}$ orthogonal to the stratum of $\mathfrak{V}$ that contains H_0. Assume, for the moment, that H_{p^*} is a differentiable point of $\mathfrak{V}$. Let $\boldsymbol{\sigma}_{\mathfrak{V}}$ be an orthonormal basis of the tangent space of $\mathfrak{V}$ at H_{p^*}. Complete this basis to an orthonormal basis $\{\boldsymbol{e}_{\mathfrak{V}}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ of $\mathcal{H}(N)$. Then the coordinates of $(H_{P^*} - H_0)$ relative to $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ are the three parameters necessary to reach multiple eigenvalues. We show that such a situation is generic in $\mathfrak{V}$. Since H_{p^*} is differentiable in $\mathfrak{V}$, there exists a neighborhood $\mathcal{N}_{H_{p^*}}$ where this $\mathfrak{V}$-genericity remains valid. Since π is continuous, $\pi^{-1}(\mathcal{N}_{H_{p^*}})$ is a neighborhood of H_0 where the differentiability of the projection holds; hence $\mathfrak{V}$-genericity. If H_{p^*} is a singular point, it belongs to a manifold $\mathfrak{M}_{n_1,n_2,\ldots,n_{\bar{N}}}$ of the algebraic variety $\mathfrak{V}$. Construct a basis $\boldsymbol{\sigma}_{\mathfrak{M}}$ of the tangent space, and complete it to a basis of $\mathcal{H}(N)$, viz., $\boldsymbol{\sigma}_{\mathfrak{M}}, \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{\left(\sum_{k=1}^{\bar{N}} n_k^2\right)-\bar{N}}$. Clearly, $\left(\sum_{k=1}^{\bar{N}} n_k^2\right) - \bar{N} \geq 3$ parameters are needed, with equality only if a unique k^* $n_{k^*} = 1$ while for all other $n_k = 1$ (see [11, p. 162] for the details regarding that last inequality). ■

Proof of Corollary 3. Setting $\mu_j = 0$ in Eq. (50), $\{\mathcal{W}_{\mu_j=0}\}$ becomes the kernel of Q and $\ker(Q) = \oplus_{i \in I(j)} \{\bar{\mathcal{V}}_{\lambda_{I(j)}}\}$. Therefore, the $\ker(Q)$ is made up of some invariant subspaces of P and if one such invariant subspace has corresponding eigenvalue $\neq 0$, the two kernels are not coincidental. ■

Note regarding end of Sec. IV-B. We prove that a diagonal Δ_f captures $\|T^u_{\boldsymbol{\zeta}_1, \boldsymbol{\upsilon}_{u,1}}\|$. Superscript and subscripts are dropped for ease of notation. Consider

$$1/\min_{\Delta_{ii}} \{\| \operatorname{diag}\{\Delta_{ii}\}\| : \det(I + \operatorname{diag}\{T_{ii}\} \operatorname{diag}\{\Delta_{ii}\} = 0\}. \tag{53}$$

Assume $\min_{\Delta_{ii}}$ is achieved for $\hat{\Delta}_{ii}$ and that $\| \operatorname{diag}\{\hat{\Delta}_{ii}\}\|$ is achieved for $|\hat{\Delta}_{oo}|$. It is claimed that $1 + T_{oo}\hat{\Delta}_{oo} = 0$. Assume by contradiction that $1 + T_{jj}\hat{\Delta}_{jj} = 0$ for $j \neq o$. Then

$$\det(I + \operatorname{diag}\{T_{ii}\} \operatorname{diag}\{\hat{\Delta}_{jj} \ldots \hat{\Delta}_{jj}\}) = 0$$

while

$$\| \operatorname{diag}\{\hat{\Delta}_{jj} \ldots \hat{\Delta}_{jj}\}\| = |\hat{\Delta}_{jj}| < |\hat{\Delta}_{oo}|,$$

which is a contradiction to the optimization in Eq. (53); hence, $1 + T_{oo}\hat{\Delta}_{oo} = 0$. Moreover, $1 + T_{ii}\hat{\Delta}_{ii} = 0$, $\forall i \neq o$. Hence, $\hat{\Delta}_{oo}$ is the minimum destabilizing perturbation and further Eq. (53) equals $|T_{oo}| = \| \operatorname{diag}\{T_{ii}\}\|$. ■

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