Relationships Between Linear Dynamically Varying Systems and Jump Linear Systems*

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May 27, 2003

Abstract

The connection between linear dynamically varying (LDV) systems and jump linear (JL) systems is explored. LDV systems have been used to model the error in nonlinear tracking problems. Some nonlinear systems, for example Axiom A systems, admit Markov partitions and their dynamics can be quantized as Markov chains. In this case, the tracking error can be approximated by a jump linear system. It is shown that (i) jump linear controllers for arbitrarily fine partitions exist if and only if the LDV controller exists; (ii) the jump linear controller stabilizes the nonlinear dynamical system; (iii) jump linear controllers provide approximations of the LDV controller. Finally, it is shown that this process is robust against such errors in the probability structure as the inaccurate assumption that an easily constructed partition is Markov.

Keywords: Linear Parameter Varying (LPV) systems, Linear Dynamically Varying (LDV) systems, Jump Linear systems, Markov partitions.

1 Introduction

Recently, robust control theory has focused on linear parametrically varying (LPV) systems [A], [AGB], [BP], [GAC], [WYPB]. When there is an underlying, at least partial, knowledge of the dynamics of the parameters, LPV systems specialize to jump linear (JL) systems [JC2], [JCFL], [CF], [FRP], [FB] and linear dynamically varying (LDV) systems [BJ1], [BJ2], [BJ3], [BJ4], [JB]. In this paper, we explore the hitherto hidden links between the last two systems.

The kind of systems considered here natually arise in the following generic nonlinear tracking problem: Find a $u \in l_2$ such that $\|\varphi(k) - \theta(k)\| \to 0$ where

$$\begin{aligned} \varphi(k+1) &= f(\varphi(k), u(k)), \\ \theta(k+1) &= f(\theta(k), 0). \end{aligned} \tag{1}$$

Defining the tracking error to be $x(k) = \varphi(k) - \theta(k)$ yields the linearized error dynamics

$$\begin{aligned} x(k+1) &= A_{\theta(k)} x(k) + B_{\theta(k)} u(k), \\ \theta(k+1) &= f(\theta(k), 0), \end{aligned} \tag{2}$$

where $A_\theta = \frac{\partial f}{\partial \theta}(\theta, 0)$ and $B_\theta = \frac{\partial f}{\partial u}(\theta, 0)$. If f is completely known, then (2) is an LDV system as discussed in Section 2. If f is not completely known, but has a symbolic description in terms of a Markov chain,

*This research was supported by National Science Foundation grant ECS-98-02594

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then system (2) becomes a jump linear system. This type of approximations is detailed in Section 4. As discussed in Section 4.3, a long standing and significant problem with the jump linear approach is that it cannot be directly shown that the jump linear controller will stabilize the nonlinear system (1). However, this paper will show that, under some conditions, the jump linear controller is an arbitrarily close approximation of the LDV controller (Theorem 11). Since the LDV controller stabilizes the nonlinear system (1), if the approximation of the LDV system is good enough, the jump linear system will also stabilize the nonlinear system (1).

Of course, not every dynamical system admits a symbolic dynamics representation. In this case, the Markovian assumption necessary for the jump linear approach cannot be met. Furthermore, even in the case where a dynamical system does have a symbolic dynamics representation, it may be extremely difficult to construct the Markov partition necessary to define the symbolic dynamics. However, as will be shown, the LDV-JL relationship still holds in these cases. Specifically, even if an arbitrary partition is used and is incorrectly assumed to be Markov (i.e., the probability of jumping from on cell to another is independent of the past jumps), then the jump linear controller may still be an approximation of the LDV controller. Moreover, in certain situations, there is little difference between the optimal jump linear controller under the incorrect Markovian assumption and the optimal controller under the correct non-Markovian assumption. Therefore, while it clearly makes sense to approximate a nonlinear system by a jump linear system when a Markov partition is known, this work shows that, even if the partition is not Markov, the jump linear approximation is still a valid approach. In this sense, this paper greatly expands the applicability of jump linear control systems.

Before detailing the connection between jump linear and LDV systems in Section 5, LDV systems are introduced (Section 2), jump linear systems reviewed (Section 4), and the semi-conjucacy between dynamical systems and symbolic systems is briefly discussed (Section 3). Section 6 provides the example of the Hénon map.

2 LDV systems

We consider a slight generalization of (2) and define an LDV system as

$$\begin{aligned} x(k+1) &= A^{LDV}_{\theta(k)} x(k) + B^{LDV}_{\theta(k)} u(k), \\ z(k) &= \begin{bmatrix} C^{LDV}_{\theta(k)} x(k) \\ D^{LDV}_{\theta(k)} u(k) \end{bmatrix}, \\ \theta(k+1) &= f(\theta(k)), \end{aligned} \tag{3}$$

where $A^{LDV}:\Theta \to \mathbb{R}^{n\times n}$, $B^{LDV}:\Theta \to \mathbb{R}^{n\times m}$, $C^{LDV}:\Theta \to \mathbb{R}^{p\times n}$, $D^{LDV}:\Theta \to \mathbb{R}^{q\times m}$ and $f:\Theta \to \Theta$, with $f \in C^0$ and Θ compact. If the maps $A,B,C,D \in C^0$, then system (2) is a continuous LDV system. The connection with a linear approximation of the nonlinear tracking system (1) can be recovered under the condition that $f \in C^1$ and setting $x := \varphi - \theta$, $A_\theta := \frac{\partial f}{\partial x}(\theta, 0)$, and $B_\theta := \frac{\partial f}{\partial u}(\theta, 0)$.

We say that the pair (A^{LDV}, f) is *exponentially stable* if system (3) is exponentially stable, that is, for $u = 0$ and $\theta_o \in \Theta$, there exists an $\alpha(\theta_o) < 1$ and a $\beta(\theta_o) < \infty$ such that, if $\theta(0) = \theta_o$, then $\|x(k)\| < \beta(\theta_o)\,\alpha(\theta_o)^k \|x(0)\|$. Furthermore, the pair (A^{LDV}, f) is *uniformly exponentially stable* if α and β can be chosen independently of $\theta(0)$. The triple (A^{LDV}, B^{LDV}, f) is *stabilizable* if there exists a bounded feedback $F:\Theta \to \mathbb{R}^{m\times n}$ such that $(A^{LDV} + B^{LDV}F, f)$ is exponentially stable. Note that uniform exponential stability is not required for a system to be stabilizable. The triple (A^{LDV}, C^{LDV}, f) is *uniformly detectable* if there is a uniformly bounded map $H^{LDV}:\Theta \to \mathbb{R}^{n\times p}$ such that $(A^{LDV} + H^{LDV}C^{LDV}, f)$ is uniformly exponentially stable. That is, there exist $\alpha_d < 1$ and $\beta_d < \infty$ such that, for all $\theta(0) \in \Theta$, we have $\|x(k)\| < \beta_d \alpha_d^k \|x(0)\|$, where $x(k+1) = \left(A^{LDV}_{f^k(\theta_o)} + H^{LDV}_{f^k(\theta_o)} C^{LDV}_{f^k(\theta_o)}\right) x(k)$.

We say that the LDV system $\left(\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial u}, f\right)$ is the LDV system induced by f. It was shown in [BJ2] that, if the LDV system (3) induced by f is uniformly exponentially stabilized by the control $u(k) = F_{\theta(k)} x(k)$, then the nonlinear system (1) with control $u(k) = F_{\theta(k)} x(k)$ is locally uniformly exponentially stable. By definition, *locally uniformly exponentially stable* means that there exist $\alpha < 1$, $\beta < \infty$, and $\gamma > 0$ such that, if $\|x(0)\| = \|\varphi(0) - \theta(0)\| < \gamma$, then $\|x(k)\| < \beta\alpha^k \|x(0)\|$, where α, β and γ can be taken independent of the initial condition θ_o, i.e. uniformly in θ_o and locally in x. Therefore, we say that the dynamical system f is LDV stabilizable if the LDV system induced by f is stabilizable.

One of the main results from [BJ2] is the following:

Theorem 1 *Suppose (3) is a continuous, uniformly detectable LDV system with* $D_\theta^{LDV\prime} D_\theta^{LDV} > 0$ *for all* $\theta \in \Theta$. *Then system (3) is LDV stabilizable if and only if there exists a bounded function* $X : \Theta \to \mathbb{R}^{n \times n}$ *with* $X_\theta' = X_\theta \geq 0$ *that satisfies the functional discrete-time algebraic Riccati equation*

$$\begin{aligned} X_\theta &= A_\theta^{LDV\prime} X_{f(\theta)} A_\theta^{LDV} + C_\theta^{LDV\prime} C_\theta^{LDV} \\ &\quad - A_\theta^{LDV\prime} X_{f(\theta)} B_\theta^{LDV} \left(D_\theta^{LDV\prime} D_\theta^{LDV} + B_\theta^{LDV\prime} X_{f(\theta)} B_\theta^{LDV} \right)^{-1} B_\theta^{LDV\prime} X_{f(\theta)} A_\theta^{LDV}. \end{aligned} \tag{4}$$

In this case, the control

$$\begin{aligned} u^{LDV}(k) &= F_{\theta(k)}^{LDV} x(k) \\ &= -\left(D_{\theta(k)}^{LDV\prime} D_{\theta(k)}^{LDV} + B_{\theta(k)}^{LDV\prime} X_{f(\theta(k))} B_{\theta(k)}^{LDV} \right)^{-1} \times B_{\theta(k)}^{LDV\prime} X_{f(\theta(k))} A_{\theta(k)}^{LDV} x(k) \end{aligned} \tag{5}$$

is optimal in the sense that it minimizes the quadratic cost

$$V(\theta_o, u, x_o) = \sum_{k=0}^{\infty} \left\| C_{f^k(\theta_o)}^{LDV} x(k) \right\|^2 + \left\| D_{f^k(\theta_o)}^{LDV} u(k) \right\|^2 .$$

Furthermore, this control uniformly exponentially stabilizes the LDV system and $x_o' X_{\theta_o} x_o = \min_u V(\theta_o, u, x_o)$ *and* X *is a continuous function.*

Remark 2 *If* f *is invertible, the uniform detectability can be weakened to detectability, which is the dual of stabilizability (see [BJ2] for details).*

Remark 3 *See [BJ2] for a discussion of numerical methods for solving (4).*

3 State partitions for dynamical systems

Symbolic dynamics on a Markov partition provides a rigorous way of describing, in a coarse way, a nonlinear dynamical system. The relationship between dynamical systems and symbolic dynamics was developed by Sinai for Anosov diffeomorphisms [S] and by Bowen for more general hyperbolic systems [B]. This relationship has been extended to other situations; for example, nonuniform hyperbolic systems [KH], expanding homeomorphisms [AH], and systems that satisfy a local product structure [R]. While there is much to be said about Markov partitions (see, e.g., [KH], [M], [P]), we only mention those aspects that relate to dropping the Markov property as will be done in the sequel.

Let $f : \Theta \to \Theta$ be a dynamical system where Θ is compact. Let $\mathcal{R} = \{R_1, R_2, \cdots, R_M\}$ be a partition of Θ, i.e., $\bigcup_{i=1}^N R_i = \Theta$ and $R_i \cap R_j = \partial R_i \cap \partial R_j$ for $i \neq j$. Define the transition matrix $T = [t_{i,j}]$ with $t_{i,j} \in \{0, 1\}$ such that $t_{i,j} = 1$ iff $f(R_i) \cap R_j \neq \emptyset$. The set of allowable bilateral and unilateral sequences is defined as

$$\Sigma_T = \left\{ s : \mathbb{Z} \to \{1, 2, \cdots, M\} : t_{s(k), s(k+1)} = 1, \ \forall k \in \mathbb{Z} \right\}, \tag{6}$$

$$\Sigma_T^+ = \left\{ s : \mathbb{N} \to \{1, 2, \cdots, M\} : t_{s(k), s(k+1)} = 1, \ \forall k \in \mathbb{N} \right\}, \tag{7}$$

from which the symbolic dynamics $\sigma^{(+)} : \Sigma_T^{(+)} \to \Sigma_T^{(+)}$ is defined by $\sigma^{(+)}(s)(k) = s(k+1)$. If the map f is among those quoted in the preceding paragraph, then, under an appropriate *Markov* partitioning $\mathcal{R}$, there exists a *semi-congugacy* h between the continuous and the symbolic dynamics, that is, the diagram

$$\begin{array}{ccc} \Sigma_T^{(+)} & \overset{\sigma^{(+)}}{\to} & \Sigma_T^{(+)} \\ \downarrow h & & \downarrow h \\ \Theta & \overset{f}{\to} & \Theta \end{array}$$

commutes; furthermore, an f-invariant measure yields a σ-invariant measure P that has the Markov property, that is,

$$P\left(s\left(k+1\right) =j|\, s\left(k\right) =i,s\left(k-1\right) =l_{1},s\left(k-2\right) =l_{2},\cdots \right) =:p_{i,j} \tag{8}$$

(See [M, Th. 9.6],[KH, Sec. 18.7] for the unilateral case and [K, Sec. 1.2] for the bilateral case.)

In case the partition $\mathcal{R}$ is not Markov, the continuous dynamics and the symbolic dynamics are no longer semi-conjugate. To derive an invariant measure P on the symbolic dynamics in a general setting, observe that, even when $\mathcal{R}$ is not Markov, $\Sigma_T^{(+)}$ equipped with the usual cylinder topology [KH, Sec. 1.9] is still compact [KH, Sec. 1.9] and that $\sigma^{(+)}$ is still continuous [KH, Sec. 1.9]. Therefore, by the Kryloff-Bogoliuboff theorem [NS, Chap. VI, Th. 9.05], there exists an invariant measure P on the symbolic dynamics defined as $P(\sigma(E)) = P(E)$ in the bilateral case and as $P((\sigma^{+})^{-1}(E)) = P(E)$ for the unilateral case, where $E \in \mathcal{B}(\Sigma_T^{+})$, the Borel σ-algebra defined to be generated by the cylinders [K, Sec. 6.2]. The only problem is that this measure *need not* be Markov. However, P has a so-called *Markov approximation* [K, p. 235], with which we will be working in the sequel.

(Another way to associate a Markov measure with a symbolic dynamics is provided in [K, Sec. 6.2].)

4 Jump linear systems

4.1 LDV system inducing a jump linear system

We now show how an LDV system may give rise to a linear system with its parameters varying according to a Markov chain, that is, a jump linear system. As described above in Section 3, depending on the dynamical system f, there may exist a Markov partition and the dynamics of f can be described by a Markov chain $s(k)$ on the finite set of symbols $\{1,2,\cdots,M\}$ with transition probabilities $p_{i,j}$. This leads to a jump linear system as follows: For each cell R_i of the Markov partition $\mathcal{R} = \{R_1, R_2, \cdots, R_M\}$, define a point $\phi_i \in int(R_i)$ for $1 \leq i \leq M$. Set

$$A_{s(k)}^{JL} = A_{\phi_{s(k)}}^{LDV} = \frac{\partial f}{\partial x}\left(\phi_{s(k)},0\right),$$
$$B_{s(k)}^{JL} = B_{\phi_{s(k)}}^{LDV} = \frac{\partial f}{\partial u}\left(\phi_{s(k)},0\right).$$

Then $A_{s(k)}^{JL}$ is a Markov chain which takes values in $\left\{A_{\phi_1}^{LDV}, A_{\phi_2}^{LDV}, \cdots, A_{\phi_M}^{LDV}\right\}$ and $B_{s(k)}^{JL}$ is a Markov chain which takes values in $\left\{B_{\phi_1}^{LDV}, B_{\phi_2}^{LDV}, \cdots, B_{\phi_M}^{LDV}\right\}$. Thus we have the jump linear system

$$x^{JL}\left(k+1\right) = A_{s(k)}^{JL} x^{JL}\left(k\right) + B_{s(k)}^{JL} u\left(k\right) \tag{9}$$

with transition probabilities given by (8). In this case, we say that the jump linear system (9) is induced by f and the partition $\mathcal{R}$. Note that, if $\max_i\left(diam\left(R_i\right)\right)$ is small and $h\left(s\right) = \theta\left(0\right)$, then $A_{s(k)}^{JL} \approx A_{\theta(k)}^{LDV}$ and $B_{s(k)}^{JL} \approx B_{\theta(k)}^{LDV}$, and therefore, $x^{JL}\left(k\right) \approx x^{LDV}\left(k\right)$. Hence, the JL system approximates the LDV system. The smaller the size of the cells R_i, the better the approximation and, as $\max_i\left(diam\left(R_i\right)\right) \to 0$ and fixed k, we have $x^{JL}\left(k\right) \to x^{LDV}\left(k\right)$.

4.2 Formal jump linear theory

We now consider a slight generalization of (9) and formally define a JL system as:

$$x\left(k+1\right) = A_{s(k)}^{JL} x\left(k\right) + B_{s(k)}^{JL} u\left(k\right), \tag{10}$$
$$z\left(k\right) = \begin{bmatrix} C_{s(k)}^{JL} x\left(k\right) \\ D_{s(k)}^{JL} u\left(k\right) \end{bmatrix},$$

where $s\left(k\right)$ is a Markov chain that takes values in a finite set $\{1,2,\cdots,M\}$ with transition probabilities $p_{i,j}$. (In [CF], $s\left(k\right)$ is allowed to take values in the countable set $\{1,2,\cdots\}$). Thus the parameters

$A^{JL}, B^{JL}, C^{JL}, D^{JL}$ are matrix-valued Markov chains. At time k, it is assumed that only $s(k)$ and $x(k)$ are known.

We say that system (10) is *stochastically stabilizable* if there exists a function $F^{JL} : \{1, 2, \cdots, M\} \to \mathbb{R}^{n\times m}$ such that the closed-loop jump linear system

$$x(k+1) = \left(A^{JL}_{s(k)} + B^{JL}_{s(k)} F^{JL}_{s(k)}\right) x(k)$$

is stochastically stable, where stochastically stable means that there exist an $\alpha < 1$ and a $\beta < \infty$ such that, for $1 \le i \le M$,

$$E\left(\|x(k)\| | \, s(0) = i\right) < \beta\alpha^k \|x(0)\|.$$

Similarly, we say that system (10) is *stochastically detectable* if there exists a function $H^{JL} : \{1, 2, \cdots, M\} \to \mathbb{R}^{n\times p}$ such that the closed-loop jump linear system

$$x(k+1) = \left(A^{JL}_{s(k)} + H^{JL}_{s(k)} C^{JL}_{s(k)}\right) x(k)$$

is stochastically stable, i.e. there exist $\alpha_d < 1$ and $\beta_d < \infty$ such that $E\left(\|x(k)\| | \, s(0) = i\right) < \beta_d \alpha_d^k \|x(0)\|$. Note that stochastic detectability is stronger than existence of a time-varying asymptotic observer.

The following is shown in [JC2], [JCFL], [CF]:

Theorem 4 *Assume that (10) is a stochastically detectable JL system and that* $\left(D^{JL\prime}_{s(k)} D^{JL}_{s(k)}\right)$ *is invertible. Then the system is stochastically stabilizable iff there exists a function* $Y : \{1, 2, \cdots, M\} \to \mathbb{R}^{n\times n}$ *with* $Y_i' = Y_i \ge 0$ *that solves the system of coupled Riccati equations*

$$\begin{aligned} Y_{s(k)} &= A^{JL\prime}_{s(k)} \hat{Y}_{s(k+1)|s(k)} A^{JL}_{s(k)} + C^{JL\prime}_{s(k)} C^{JL}_{s(k)} \\ &- A^{JL\prime}_{s(k)} \hat{Y}_{s(k+1)|s(k)} B^{JL}_{s(k)} \left(D^{JL\prime}_{s(k)} D^{JL}_{s(k)} + B^{JL\prime}_{s(k)} \hat{Y}_{s(k+1)|s(k)} B^{JL}_{s(k)}\right)^{-1} B^{JL\prime}_{s(k)} \hat{Y}_{s(k+1)|s(k)} A^{JL}_{s(k)}, \end{aligned} \tag{11}$$

where

$$\hat{Y}_{s(k+1)|s(k)} = E\left(Y_{s(k+1)} \middle| \, s(k)\right) = \sum_{j=1}^{M} p_{s(k),j} Y_j.$$

In this case, the control

$$\begin{aligned} u(k) &:= F^{JL}_{s(k)} x(k) \\ &:= -\left(D^{JL\prime}_{s(k)} D^{JL}_{s(k)} + B^{JL\prime}_{s(k)} \hat{Y}_{s(k+1)|s(k)} B^{JL}_{s(k)}\right)^{-1} B^{JL\prime}_{s(k)} \hat{Y}_{s(k+1)|s(k)} A^{JL}_{s(k)} x(k) \end{aligned} \tag{12}$$

is optimal in the sense that

$$u = \arg\min_{u\in U_{JL}} E\left(\sum_{k=0}^{\infty} \left\|C^{JL}_{s(k)} x(k)\right\|^2 + \left\|D^{JL}_{s(k)} u(k)\right\|^2\right), \tag{13}$$

where U_{JL} *is the set of* u *such that* $u(k) \in \mathcal{F}_k$*, where* $\mathcal{F}_k$ *denotes the* σ*-algebra generated by* $s(l)$ *and* $x(l)$, $l \le k$. *Furthermore, the control (12) is stochastically stabilizing and*

$$x_o' Y_{s(0)} x_o = \min_{u\in U_{JL}} E\left(\sum_{k=0}^{\infty} \left\|C^{JL}_{s(k)} x(k)\right\|^2 + \left\|D^{JL}_{s(k)} u(k)\right\|^2 |s(0)\right).$$

The analogue of the following lemma is a standard fact for time-varying systems. Proofs can be found in the literature; for example, in [CF].

Lemma 5 *Assume the jump linear system (10) is stochastically stabilizable and detectable and* $D_i^{JL\prime} D_i^{JL} > 0$ *for all* i. *Assume that* $Y \ge 0$ *is the solution to the coupled Riccati equations (11). Furthermore, assume there is a* $\bar{Y} < \infty$ *such that* $\|Y_i\| \le \bar{Y}$ *for* $1 \le i \le M$. *In this case,* $E\left(\|x(k)\|^2 \middle| \mathcal{F}_l\right) \le \beta\alpha^{k-l} \|x(l)\|^2$ *for* $k \ge l$ *where* α *and* β *can be taken to depend only on* α_d *and* β_d *in the definition of stochastic detectability, and on the bound* $\bar{Y}$.

4.3 Jump linear control of nonlinear systems

Consider the system (2) and suppose that $f \in C^1$ induces Markov partitions with arbitrarily small rectangles. In such a situation, it is customary to approximate the nonlinear dynamically varying parameters with a Markov chain running on a fine Markov partition, design the resulting JL controller F^{JL}, and use this controller to achieve the nonlinear tracking for system (1). We show that this jump linear approach is more difficult than it might first appear.

Applying the jump linear controller F^{JL} to system (1) yields

$$x(k+1) = \left(A^{JL}_{s(k)} + B^{JL}_{s(k)} F^{JL}_{s(k)}\right) x(k) + \eta\left(\theta(k), x(k), F^{JL}_{s(k)} x(k)\right), \tag{14}$$

where $x = \varphi - \theta$ and η accounts for errors due to linearization and quantization. Since $f \in C^1$, it is not hard to show that, if $x(k)$ is small and the partition fine, then $\eta\left(\theta(k), x(k), F^{JL}_{s(k)} x(k)\right)$ is small. From this observation, using techniques similar to [BJ2], [BJ3], it is possible to show that the system is locally stochastically stable; that is, if $\|x(0)\|$ is small, $E(\|x(k)\|) \to 0$ as $k \to \infty$. The problem is that there exists a nonzero probability that $x(k) \not\to 0$ as $k \to \infty$.

In fact, it is not hard to find examples where $\eta(x) = O(||x||)$, the system is stochastically stable without the η term, but there exists a non-zero probability that the system with this nonlinearity is unstable. For example, consider $x(k+1) = a_i x(k) + \eta(x)$ with $a_1 = 0.1$ and $a_2 = 2$ with transition probabilities $[p_{i,j}] = \begin{bmatrix} 0.9 & 0.1 \\ 0.9 & 0.1 \end{bmatrix}$, and with $\eta(x) = \begin{cases} 0 \text{ if } x < 1 \\ 10x \text{ if x} \geq 1 \end{cases}$. Without the η term, the system is easily seen to be stochastically stable (e.g., set $B = 0$ in Theorem 4). However, with the η term, for all $x(0) > 0$, there is a positive probability that $x(k_0) > 1$ for some k_0, from where the system goes unstable. Of course, using Chebyshev's inequality, one can show that by limiting $\|x(0)\|$ the closed-loop nonlinear system is stable with a probability close to one.

The difficulty is that stochastic stability implies stability over the average orbit $\{\theta(k)\}$. But, when a particular orbit is chosen, stochastic stability does not imply anything about the stability along this orbit. These difficulties can be avoided by using techniques described in [JC1]. However, the method in [JC1] appears to be overly conservative since simulations show that the standard jump linear controller works for a fine enough partition. The next section shows that the simulations are correct and that, for fine enough partitions, the jump linear controller stabilizes the nonlinear system.

5 Main results

Next, it will be shown that, if the nonlinear system is LDV stabilizable, then, for a fine enough partition, the jump linear controller stabilizes the nonlinear system (Proposition 7). Conversely, if as the partition is refined, the solution to the jump linear coupled Riccati equations (11) remains bounded, then the system is LDV stabilizable (Proposition 9). In this case, as the partition is refined, the jump linear controller approaches the LDV controller (Theorem 11). Moreover, this process is robust against errors in the Markov partitioning. Specifically, if the partition is incorrectly assumed to be Markov, then the resulting jump linear controller still approximates the LDV controller and, if the partition is fine enough, stabilizes the nonlinear system.

Once we set a partition $\mathcal{R} = \{R_1, R_2, \cdots, R_M\}$, we have a set of transition probabilities

$$P\left(s(k+1) = j \mid s(k) = i, s(k-1) = l_1, s(k-2) = l_2, \cdots\right)$$

reflecting the actual dynamics f. A path s is said to be f-admissible if along each finite section of s the transition probability is nonvanishing.

Next to the above actual probability structure, we define a *Markov approximation* ([K, p. 235])

$$p_{i,j} := P\left(s(k+1) = j \mid s(k) = i\right)$$

even though it is possible that

$$p_{i,j} \neq P\left(s(k+1) = j \mid s(k) = i, s(k-1) = l_1, s(k-2) = l_2, \cdots\right).$$

Note that, since the partition need not be Markov, the transition probabilities can, for practical purposes, be computed as

$$p_{i,j} = \frac{\#\left\{k : f^k(\theta) \in R_i,\ f^{k+1}(\theta) \in R_j\right\}}{\#\left\{k : f^k(\theta) \in R_i\right\}}, \tag{15}$$

where θ is a transitive point and assuming f is ergodic. By JL-admissible path, we mean a path with nonzero Markov transition probabilities, that is, $p_{s(k),s(k+1)} \neq 0$. Note that some JL-admissible paths may not be f-admissible. Define $E_{\mathcal{R},p}(\cdot)$ to be the expectation operator with the probabilistic structure induced by $p_{i,j}$.

With $\mathcal{R}$ chosen, define $h : \Sigma_T \to \Theta$ as in Section 3. Recall that $h\left(\{s \in \Sigma_T : s(0) = i\}\right) = R_i$. Define a quantization map

$$g : \Theta \to \{1, 2, \cdots, M\}$$

as follows: If $\theta \in R_i$ and $\theta \notin R_j$ for all $j \neq i$, then $g(\theta) = i$. If $\theta \in R_i \cap (R_j \cap \cdots \cap R_k)$, where the latter is the maximal such intersection, then we have a choice as to the value of $g(\theta) = i$ or j. By invoking the Axiom of Choice, we can define g such that $g(\theta) = i$ implies that $\theta \in R_i$.

As done in Section 4.1, for each cell R_i, define $\phi_i \in \mathrm{int}(R_i)$. Thus $\phi_{s(k)}$ is a Markov chain that takes values in $\{\phi_1, \phi_2, \cdots, \phi_M\}$. Since $R_i \cap R_j = \partial R_i \cap \partial R_j$, we have $g(\phi_i) = i$. This suggests the notation

$$\begin{aligned} \tilde{g}^{-1} &: \{1, 2, \cdots, M\} \to \Theta, \\ \tilde{g}^{-1}(i) &= \phi_i \end{aligned}$$

and the definitions

$$\begin{aligned} A_i^{JL} &:= A_{\tilde{g}^{-1}(i)}^{LDV}, \qquad & B_i^{JL} &:= B_{\tilde{g}^{-1}(i)}^{LDV}, \\ C_i^{JL} &:= C_{\tilde{g}^{-1}(i)}^{LDV}, \qquad & D_i^{JL} &:= D_{\tilde{g}^{-1}(i)}^{LDV}. \end{aligned}$$

Clearly, if the mesh of the partition is small, i.e. $mesh(\mathcal{R}) = \max_i \left(diam(R_i)\right)$ is small, then $\tilde{g}^{-1}(g(\theta)) \approx \theta$ and

$$\begin{aligned} A_\theta^{LDV} &\approx A_{g(\theta)}^{JL}, \qquad & B_\theta^{LDV} &\approx B_{g(\theta)}^{JL}, \\ C_\theta^{LDV} &\approx C_{g(\theta)}^{JL}, \qquad & D_\theta^{LDV} &\approx D_{g(\theta)}^{JL}. \end{aligned}$$

Likewise, if k is finite and $mesh(\mathcal{R})$ is small, then $f^k(\theta) \approx f^k\left(\tilde{g}^{-1}(g(\theta))\right)$. Note that the functions h, g and $\tilde{g}^{-1}$ depend on the partition $\mathcal{R}$. Thus these functions should be written $h_\mathcal{R}$, $g_\mathcal{R}$ and $\tilde{g}_\mathcal{R}^{-1}$. However, to reduce clutter, the dependence on the partition is dropped.

Define

$$supp\left(p_{g(\theta),g(\cdot)}\right) = \overline{\left\{\varphi \in \Theta : p_{g(\theta),g(\varphi)} \neq 0\right\}}$$

and define $supp\left(p_{g(\theta),g(\cdot)}^k\right)$ similarly, where $p_{i,j}^k$ is the i,j element of the k^{th} power of the matrix $[p_{i,j}]$.

Lemma 6 *For fixed $k < \infty$, as $mesh(\mathcal{R}) \to 0$,*

$$diam\left(supp\left(p_{g(\theta),g(\cdot)}^k\right)\right) \to 0$$

uniformly in θ.

Proof. Set $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ such that, if $mesh(\mathcal{R}) < \delta$, then for each $\theta \in \Theta$, $diam\left(f\left(\{\varphi \in \Theta : g(\theta) = g(\varphi)\}\right)\right) \leq diam\left(f\left(R_{g(\theta)}\right)\right) < \varepsilon$. Thus $diam\left(supp\left(p_{g(\theta),g(\cdot)}\right)\right) < \varepsilon + 2\delta$, where the 2δ term is due to points x such that $x \notin f\left(R_{g(\theta)}\right)$, but $g(x) = g(\varphi)$ and $\varphi \in f\left(R_{g(\theta)}\right)$, so $\|x - \varphi\| \leq mesh(R) < \delta$. (Note that the 2δ is not needed in case of the Markov partitioning of an expanding endomorphism [M, Chap. IV, Ex. 9.1], [K, Sec. 1.2].) Hence, the lemma holds for $k = 1$. Since k is finite, the same reasoning can be applied k times.■

Proposition 7 *Assume that A^{LDV}, B^{LDV} are continuous and that (A^{LDV}, B^{LDV}, f) is LDV stabilizable. In this case, there exists a $\delta > 0$ such that, if $mesh(\mathcal{R}) < \delta$, then the jump linear system induced by f and $\mathcal{R}$ is stochastically stabilizable. Furthermore, assuming that $C^{LDV} : \Theta \to \mathbb{R}^{p\times n}$ and $D^{LDV} : \Theta \to \mathbb{R}^{q\times m}$ are continuous and $D^{LDV\prime}D^{LDV} > 0$, if $mesh(\mathcal{R}) < \delta$, then there is a bound $\bar{Y}$ on the solution Y to the coupled Riccati equations which is independent of the partition.*

Proof. Let F^{LDV} be the optimal LDV feedback. Define $F(k, s(0)) := F^{LDV}_{f^k(\tilde{g}^{-1}(s(0)))}$, that is, the optimal LDV feedback assuming $\theta(0) = \tilde{g}^{-1}(s(0))$. Note that $F(k, s(0)) \in \mathcal{F}_0 \subset \mathcal{F}_k$. Thus $u(k) = F(k, s(0))\, x(k) \in U_{JL}$. Since the LDV feedback uniformly exponentially stabilizes the LDV system, for any $\epsilon > 0$, there exists an $N < \infty$ such that, for all $\theta \in \Theta$,

$$\left\| \prod_{k=0}^{N-1} \left(A^{LDV}_{f^k(\theta)} + B^{LDV}_{f^k(\theta)} F^{LDV}_{f^k(\theta)} \right) \right\| \le \frac{\epsilon}{2}. \tag{16}$$

Since the left-hand side of Equation (16) is continuous in A^{LDV} and B^{LDV}, there exists a $\gamma > 0$ such that, for any perturbed sequences $\tilde{A}_k, \tilde{B}_k$ satisfying

$$\left\| \tilde{A}_k - A^{LDV}_{f^k(\theta)} \right\|, \left\| \tilde{B}_k - B^{LDV}_{f^k(\theta)} \right\| < \gamma,$$

we have $\left\| \prod_{k=0}^{N-1} \left(\tilde{A}_k + \tilde{B}_k F^{LDV}_{f^k(\theta)} \right) \right\| \le \epsilon$. Since A^{LDV} and B^{LDV} are uniformly continuous, there exists a $\lambda > 0$ such that, if $\|\varphi - \theta\| < \lambda$, then $\|A^{LDV}_\varphi - A^{LDV}_\theta\|, \|B^{LDV}_\varphi - B^{LDV}_\theta\| < \gamma$. By Lemma 6, there exists $\delta_1 > 0$ such that if $mesh(\mathcal{R}) < \delta_1$ then $diam\left(supp\left(p^k_{g(\theta), g(\cdot)} \right) \right) < \lambda$ for all $k \le N$. Let $\mathcal{R}$ be a partition such that $mesh(\mathcal{R}) < \delta_1$ and let $\{s(k) : 0 \le k \le N-1\}$ be a JL-admissible path, i.e., $p_{s(i), s(i+1)} > 0$, starting from $s(0) = g(\theta)$. Then $\left\| \tilde{g}^{-1}(s(k)) - f^k(\theta) \right\| < \lambda$ for $k \le N$. Thus $\left\| A^{JL}_{s(k)} - A^{LDV}_{f^k(\theta)} \right\|, \left\| B^{JL}_{s(k)} - B^{LDV}_{f^k(\theta)} \right\| < \gamma$ and therefore

$$\left\| \prod_{k=0}^{N-1} \left(A^{JL}_{s(k)} + B^{JL}_{s(k)} F^{LDV}_{f^k(\theta)} \right) \right\| < \epsilon$$

and, averaging over all JL-admissible paths (including those that are not f-admissible), we get

$$E_{\mathcal{R},p} \left(\left\| \prod_{k=0}^{N-1} \left(A^{JL}_{s(k)} + B^{JL}_{s(k)} F^{LDV}_{f^k(\theta)} \right) \right\| \, \middle| \, s(0) = g(\theta) \right) < \epsilon. \tag{17}$$

Since δ_1 was chosen independently of θ, we conclude that the JL system is stochastically stabilizable for $mesh(\mathcal{R}) < \delta_1$.

With C^{LDV} and D^{LDV} given, bounds on C^{JL} and D^{JL} that do not depend on $mesh(\mathcal{R})$ can be found. Inequality (17) can be used to show that there is a $\bar{Y} < \infty$ such that, if Y solves the coupled Riccati equations (11), then for $mesh(\mathcal{R}) < \delta_1$, $\|Y_i\| < \bar{Y}$ for $1 \le i \le M$, where $\bar{Y}$ is independent of the partition. ■

The following can be proved in the same fashion as the above proposition.

Proposition 8 *Assume that $A^{LDV} : \Theta \to \mathbb{R}^{n\times n}, C^{LDV} : \Theta \to \mathbb{R}^{p\times n}$ are continuous and $\left(A^{LDV}, C^{LDV}, f\right)$ is LDV detectable. Then there exists a $\delta > 0$ such that, if $mesh(\mathcal{R}) < \delta$, the jump linear system induced by A^{LDV}, C^{LDV}, f and $\mathcal{R}$ is stochastically detectable. Furthermore, for $mesh(\mathcal{R}) < \delta$, the α_d and β_d in the definition of stochastic detectability can be taken independent of the partition.*

Now, we consider a sequence of partitions $\{\mathcal{R}^t\}$ such that $mesh(\mathcal{R}^t) \to 0$ as $t \to \infty$. Since $(A, B, C, D)^{LDV}$ is continuous, this sequence induces a well behaved sequence $(A, B, C, D)^{JL,t}$ of JL systems. For every t, $\mathcal{R}^t, f$ induce an invariant measure P^t on $(\mathcal{R}^t)^{\mathbb{Z}(\mathbb{N})}$, which in turn induces a measure on $\Theta^{\mathbb{Z}(\mathbb{N})}$. By Tychonoff's theorem [D, Chap. XI, Th. 1.4], $\Theta^{\mathbb{Z}(\mathbb{N})}$ is compact and therefore there exists a subsequence of measures that converges [NS, Chap. VI, Th. 9.04]. Such a subsequence will be called consistent, for it induces a well behaved sequence of measures. For such a consistent sequence of partitions, $\mathcal{R}^t, f$ induce, under appropriate existence conditions, a converging sequence of solutions Y^t to the coupled Riccati equations (11) and a converging sequence $F^{JL,t}$ of jump linear controllers.

Proposition 9 *Assume that* $C^{LDV}:\Theta\to\mathbb{R}^{p\times n}$, $D^{LDV}:\Theta\to\mathbb{R}^{q\times m}$ *are continuous,* $\left(A^{LDV},C^{LDV},f\right)$ *is uniformly detectable, and there exists a consistent sequence of partitions* $\{\mathcal{R}^t\}$ *with* $mesh(\mathcal{R}^t)\to 0$ *such that* $\|Y_i^t\|<\bar{Y}$ *for all* i*, where* $\{Y_i^t\}$ *is the solution set to the coupled Riccati equations (11) for the partition* $\mathcal{R}^t$*. In this case, the LDV system induced by* f *is stabilizable.*

Proof. Since $\|Y^t\|<\bar{Y}$ for all t, by Lemma 5, for all $\epsilon>0$ and all t, there exists an $N<\infty$ such that, if the optimal jump linear feedback $F_{s(k)}^{JL,t}$ is applied, we have $E\left(\|x(N)\|\,|\,s(0)\right)<\frac{\epsilon}{2}\|x(0)\|$ for all $s(0)$, where $x(k+1)=\left(A_{s(k)}^{JL}+B_{s(k)}^{JL}F_{s(k)}^{JL}\right)x(k)$. Thus, for each $s(0)=g(\theta_o)$, there exists a JL-admissible path $\{s(k):0\le k\le N\}$ with $s(0)=g(\theta_o)$ such that

$$\left\|\prod_{k=0}^{N}\left(A_{s(k)}^{JL,t}+B_{s(k)}^{JL,t}F_{s(k)}^{JL,t}\right)\right\|<\frac{\epsilon}{2}. \tag{18}$$

Since Equation (18) is uniformly continuous in $A^{JL,t}$ and $B^{JL,t}$, it is possible to show that there exists a $\delta>0$ such that, if $mesh(\mathcal{R}^t)<\delta$, then $\left\|\prod_{k=0}^{N}\left(A_{f^k(\theta_o)}^{LDV}+B_{f^k(\theta_o)}^{LDV}F_{s(k)}^{JL,t}\right)\right\|<\epsilon$. Since this applies for all θ_o, we conclude that $F^{JL,t}$ stabilizes the LDV system. ■

Combining propositions 7 and 9 yields:

Theorem 10 *Let* $C^{LDV}:\Theta\to\mathbb{R}^{p\times n}$ *and* $D^{LDV}:\Theta\to\mathbb{R}^{q\times m}$ *be continuous with* $D_\theta^{LDV\prime}D_\theta^{LDV}>0$ *and let* $\left(A^{LDV},C^{LDV},f\right)$ *be uniformly detectable. The LDV system induced by* f *is stabilizable if and only if, for any consistent sequence of partitions* $\mathcal{R}^t$ *such that* $mesh(\mathcal{R}^t)\to 0$*, the Markov jump linear systems induced by* f *and* $\mathcal{R}^t$ *are stochastically stabilizable with bounded optimal quadratic costs, where the bound does not depend on* t*.*

Thus the existence of a stabilizing LDV controller is linked to the existence of a series of stabilizing jump linear controllers. Now we show that actually these controllers are nearly identical.

Theorem 11 *Let* $C^{LDV}:\Theta\to\mathbb{R}^{p\times n}$ *and* $D^{LDV}:\Theta\to\mathbb{R}^{q\times m}$ *be continuous with* $D_\theta^{LDV\prime}D_\theta^{LDV}>0$ *and let* $\left(A^{LDV},C^{LDV},f\right)$ *be uniformly detectable. Assume that* f *is LDV stabilizable or, equivalently, assume that there exists a bounded sequence of solutions* Y^t *to the coupled Riccati equations (11) associated with a consistent sequence of partitions with* $mesh(\mathcal{R}^t)\to 0$*. Then*

$$\sup_{\theta\in\Theta}\left\|X_\theta-Y_{g(\theta)}^t\right\|\to 0 \text{ as } t\to\infty,$$

where X *solves the functional Riccati equation and* Y^t *solves the coupled Riccati equations.*

Proof. Set $\varepsilon>0$. First we show that there exists a $\delta_1>0$ such that, if $mesh(\mathcal{R}^t)<\delta_1$, then $Y_{g(\theta)}^t\le X_\theta+\varepsilon I$. Consider the optimal LDV feedback assuming $\theta(0)=\tilde{g}^{-1}(s(0))$. As in the proof of Proposition 7, this feedback $F_{f^k(\tilde{g}^{-1}(s(0)))}^{LDV}\in\mathcal{F}_0\subset\mathcal{F}_k$. Thus $u(k)=F_{f^k(\tilde{g}^{-1}(s(0)))}^{LDV}x(k)\in U_{JL}$. Set $\|x_o\|=1$ and $\theta=\tilde{g}^{-1}(s(0))$. Let F denote any feedback sequence. Define $x^{LDV}(F,k+1)=\left(A_{f^k(\theta)}^{LDV}+B_{f^k(\theta)}^{LDV}F_k\right)x^{LDV}(F,k)$ with $x^{LDV}(F,0)=x_o$. Likewise, define $x^{JL}(F,k+1)=\left(A_{s(k)}^{JL}+B_{s(k)}^{JL}F_k\right)x^{JL}(F,k)$ with $x^{JL}(F,0)=x_o$. Since the LDV feedback is exponentially stabilizing, there exists an $N<\infty$ such that, for all $\theta\in\Theta$,

$$\left\|x^{LDV}\left(F^{LDV},N\right)\right\|^2<\frac{\varepsilon}{4\overline{Y}},$$

where $\overline{Y}$ is the bound on the solution to the JL coupled Riccati equations. Since $x^{LDV}\left(F^{LDV},N\right)$ is uniformly continuous in A^{LDV} and B^{LDV} and by Lemma 6, there exists a $\delta_1>0$ such that, if $mesh(\mathcal{R})<\delta_1$, then, for each JL-admissible path $\left\{s:s(0)=g(\theta),p_{s(k),s(k+1)}>0\right\}$,

$$\left\|x^{JL}\left(\tilde{F},N\right)\right\|^2<\frac{\varepsilon}{2\overline{Y}}, \tag{19}$$

where $\tilde{F}_k := F^{LDV}_{f^k(\theta)} = F^{LDV}_{f^k(\tilde{g}^{-1}(s(0)))}$. Hence,

$$E_{\mathcal{R},p}\left(\left\|x^{JL}\left(\tilde{F},N\right)\right\|^2 \middle| s(0) = g(\theta)\right) \leq \frac{\varepsilon}{2\bar{Y}}. \tag{20}$$

Similarly, since

$$\begin{aligned}&\sum_{k=0}^{N-1}\left\|C^{LDV}_{f^k(\theta)}x^{LDV}\left(F^{LDV},k\right)\right\|^2 + \left\|D^{LDV}_{f^k(\theta)}F^{LDV}_{f^k(\theta)}x^{LDV}\left(F^{LDV},k\right)\right\|^2 \\ &= x(0)' X_\theta x(0) - x^{LDV}\left(F^{LDV},N\right)' X_{f^N(\theta)}x^{LDV}\left(F^{LDV},N\right)\end{aligned}$$

and since the left-hand side is continuous in A^{LDV}, B^{LDV}, C^{LDV}, and D^{LDV}, by an argument similar to the one that led to (19), there exists a $\delta_2 > 0$ such that, if $mesh(\mathcal{R}) < \delta_2$, then, for any JL-admissible path $\left\{s : s(0) = g(\theta), p_{s(k),s(k+1)} > 0\right\}$,

$$\begin{aligned}&\left(\sum_{k=0}^{N-1}\left\|C^{JL}_{s(k)}x^{JL}\left(\tilde{F},k\right)\right\|^2 + \left\|D^{JL}_{s(k)}F^{LDV}_{f^k(\theta)}x^{JL}\left(\tilde{F},k\right)\right\|^2\right) \\ &\leq x_o' X_\theta x_o - x^{LDV}\left(F^{LDV},N\right)' X_{f^N(\theta)}x^{LDV}\left(F^{LDV},N\right) + \frac{\varepsilon}{2}.\end{aligned} \tag{21}$$

Assume Y is the positive definite solution to the coupled Riccati equations (11), that is,

$$\begin{aligned}&x(0)' Y_{s(0)}x(0) \\ &= E_{\mathcal{R},p}\left(\sum_{k=0}^{N-1}\left\|C^{JL}_{s(k)}x^{JL}\left(F^{JL},k\right)\right\|^2 + \left\|D^{JL}_{s(k)}F^{JL}_{s(k)}x^{JL}\left(F^{JL},k\right)\right\|^2\right. \\ &\left.+x^{JL}\left(F^{JL},N\right)' Y_{s(N)}x^{JL}\left(F^{JL},N\right)\middle| s(0) = g(\theta)\right).\end{aligned}$$

Therefore, if the control $u(k) = F^{LDV}_{f^k(\tilde{g}^{-1}(s(0)))}x(k)$ is applied for $k < N$, the control $u(k) = F^{JL}_{s(k)}x(k)$ is applied for $k \geq N$, and $mesh(\mathcal{R}) < \min(\delta_1, \delta_2)$, it follows that applying (21) and (20) yields

$$\begin{aligned}&x(0)' Y_{s(0)}x(0) \\ &= E_{\mathcal{R},p}\left(\sum_{k=0}^{N-1}\left\|C^{JL}_{s(k)}x^{JL}\left(F^{JL},k\right)\right\|^2 + \left\|D^{JL}_{s(k)}F^{JL}_{s(k)}x^{JL}\left(F^{JL},k\right)\right\|^2\right. \\ &\left.+x^{JL}\left(F^{JL},N\right)' Y_{s(N)}x^{JL}\left(F^{JL},N\right)\middle| s(0) = g(\theta)\right) \\ &\leq E_{\mathcal{R},p}\left(\sum_{k=0}^{N-1}\left\|C^{JL}_{s(k)}x^{JL}\left(\tilde{F},k\right)\right\|^2 + \left\|D^{JL}_{s(k)}F^{LDV}_{f^k(\theta)}x^{JL}\left(\tilde{F},k\right)\right\|^2\right. \\ &\left.+x^{JL}\left(\tilde{F},k\right)' Y_{s(N)}x^{JL}\left(\tilde{F},k\right)\middle| s(0) = g(\theta)\right) \\ &\leq x_o' X_\theta x_o - x^{LDV}\left(F^{LDV},N\right)' X_{f^N(\theta)}x^{LDV}\left(F^{LDV},N\right) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2\bar{Y}}\bar{Y} \\ &\leq x(0)' X_\theta x(0) + \varepsilon.\end{aligned}$$

Since $\|x_o\| = 1$ and θ is arbitrary,

$$Y_{s(g(\theta))} \leq X_\theta + \varepsilon I. \tag{22}$$

Next, we show that $X_\theta \leq Y + \varepsilon I$. Since Y^t is bounded, for $mesh(\mathcal{R}^t)$ small enough, by Corollary 8 and Lemma 5, there exists an $N < \infty$ such that, for all $\theta \in \Theta$, $\|x_o\| = 1$ and with the optimal jump linear control applied,

$$E_{\mathcal{R}^t,p}\left(\left\|x^{JL}\left(F^{JL},N\right)\right\|^2 \middle| s(0) = g(\theta)\right) \leq \frac{\varepsilon}{2\bar{X}}, \tag{23}$$

where $\|X_\varphi\| \leq \bar{X} < \infty$ for all $\varphi \in \Theta$. Since F^{LDV} is the optimal control and X_θ is the minimum quadratic cost, for any sequence $\{n(l) \in \mathcal{R}^t : 0 \leq l < N\}$,

$$\begin{aligned}
& x_o' X_\theta x_o \\
& \leq \sum_{k=0}^{N-1} \left\| C_{f^k(\theta)}^{LDV} x^{LDV}\left(\hat{F}, k\right)\right\|^2 + \left\| D_{f^k(\theta)}^{LDV} \hat{F}_k x^{LDV}\left(\hat{F}, k\right)\right\|^2 \\
& + x^{LDV}\left(\hat{F}, N\right)' X_{f^N(\theta)} x^{LDV}\left(\hat{F}, N\right),
\end{aligned}$$

where $\hat{F}_k = F_{n(k)}^{JL}$. Since the right-hand side of this expression is continuous in $A^{LDV}, B^{LDV}, C^{LDV}$ and D^{LDV}, there exists a $\delta_4 > 0$ such that, if $mesh(\mathcal{R}) < \delta_4$ and if $\left\{s : s(0) = g(\theta), p_{s(k),s(k+1)} > 0\right\}$ is a JL-admissible path, then

$$\begin{aligned}
& \sum_{k=0}^{N-1} \left\| C_{f^k(\theta)}^{LDV} x^{LDV}\left(\hat{F}, k\right)\right\|^2 + \left\| D_{f^k(\theta)}^{LDV} F_{n(l)}^{JL} x^{LDV}\left(\hat{F}, k\right)\right\|^2 + x^{LDV}\left(\hat{F}, N\right)' X_{f^N(\theta)} x^{LDV}\left(\hat{F}, N\right) \\
& - \sum_{k=0}^{N-1} \left\| C_{s(l)}^{JL} x^{JL}\left(F^{JL}, k\right)\right\|^2 + \left\| D_{s(l)}^{JL} F_{n(l)}^{JL} x^{JL}\left(F^{JL}, k\right)\right\|^2 - x^{JL}\left(F^{JL}, N\right)' X_{f^N(\theta)} x^{JL}\left(F^{JL}, N\right) \\
& < \frac{\varepsilon}{2}.
\end{aligned}$$

Thus combining these last two expressions yields

$$\begin{aligned}
& x_o' X_\theta x_o \\
& \leq \sum_{k=0}^{N-1} \left\| C_{s(l)}^{JL} x^{JL}\left(F^{JL}, k\right)\right\|^2 + \left\| D_{s(l)}^{JL} F_{n(l)}^{JL} x^{JL}\left(F^{JL}, k\right)\right\|^2 \\
& + x^{JL}\left(F^{JL}, N\right)' X_{f^N(\theta)} x^{JL}\left(F^{JL}, N\right) + \frac{\varepsilon}{2}.
\end{aligned}$$

This equation remains true for any weighted sum of JL-admissible paths; in particular,

$$\begin{aligned}
x_o' X_\theta x_o & \leq E_{\mathcal{R},p}\left(\sum_{k=0}^{N-1} \left\| C_{s(k)}^{JL} x^{JL}\left(F^{JL}, k\right)\right\|^2 + \left\| D_{s(k)}^{JL} F_{s(k)}^{JL} x^{JL}\left(F^{JL}, k\right)\right\|^2\right. \\
& \left. + x^{JL}\left(F^{JL}, N\right)' X_{f^N(\theta)} x^{JL}\left(F^{JL}, N\right) | s(0) = g(\theta)\right) + \frac{\varepsilon}{2} \\
& \leq x_o' Y_{g(\theta)} x_o + E_{\mathcal{R},p}\left(x^{JL}\left(F^{JL}, N\right)' X_{f^N(\theta)} x^{JL}\left(F^{JL}, N\right) \Big| s(0) = g(\theta)\right) + \frac{\varepsilon}{2} \\
& \leq x_o' Y_{g(\theta)} x_o + \varepsilon,
\end{aligned}$$

where the last inequality follows from (23). Thus

$$X_\theta \leq Y_{g(\theta)} + \varepsilon I.$$

Combining this with (22) and $mesh(\mathcal{R}) < \min(\delta_1, \delta_2, \delta_3)$ yields, for all $\theta \in \Theta$,

$$\left\| X_\theta - Y_{g(\theta)} \right\| < \varepsilon.$$

■

6 Example

The relationship between LDV and JL controllers is illustrated through an example. We examine the Hénon system, as it is a benchmark nonlinear system with complicated dynamics. The Hénon system is defined as

$$\begin{bmatrix} \theta_1(k+1) \\ \theta_2(k+1) \end{bmatrix} = \begin{bmatrix} f_1(\theta(k), u(k)) \\ f_2(\theta(k), u(k)) \end{bmatrix} = \begin{bmatrix} 1 - (a + u(k))\,\theta_1(k)^2 + \theta_2(k) \\ b\theta_1(k) \end{bmatrix},$$

where u is the control input. In this example, $a = 1.4$ and $b = 0.3$. For these parameter values and $u \equiv 0$, it is known that the Hénon map has an attractor Θ, that is, there exists an open set $V \supsetneq \Theta$ such that $\lim_{k\to\infty} d\left(f^k\left(\theta_o\right), \Theta\right) = 0$ for all $\theta_0 \in V$. This attractor is the crescent shaped object in the horizontal (θ_1, θ_2) plane of Figure 1. The objective of the tracking problem is to define a control u such that, if $\varphi(k+1) = f(\varphi(k), u(k))$ and $\theta(k+1) = f(\theta(k), 0)$, then $\lim_{k\to\infty} \|\varphi(k) - \theta(k)\| = 0$. When designing such a controller, there are two possible approaches...

The first is approach is to model the tracking error $\varphi - \theta =: x$ as the LDV system

$$\begin{aligned} x^{LDV}(k+1) &= A^{LDV}_{\theta(k)} x^{LDV}(k) + B^{LDV}_{\theta(k)} u(k), \\ \theta(k+1) &= f(\theta(k), 0), \end{aligned}$$

where

$$A^{LDV}_{\theta} = \begin{bmatrix} -2a\theta_1 & 1 \\ b & 0 \end{bmatrix}, \; B^{LDV}_{\theta} = \begin{bmatrix} -2\theta_1 \\ 0 \end{bmatrix}.$$

An LDV controller can be found by solving the functional algebraic Riccati equation (4). In [BJ2], a few different approaches for solving (4) are developed.

A second approach to designing a tracking controller is to develop a jump linear model of the error. The first step in defining a jump linear tracking error model is to set a partition of the state space Θ. Since the jump linear model assumes that the system parameters vary according to a Markov chain, it would be best if a Markov partition could be found. However, in the case of the Hénon map, it is not known whether such a partition exists. Even in the case a Markov partition would be known to exist, it could be very difficult to construct it. However, as shown in the main body of this paper, a stabilizing controller can be obtained even though the symbolic dynamics of the parameters is constructed on a partition that is *not* Markov. A partition can be simply defined as

$$\mathcal{R}^t = \left\{ R^t_{s=(s_1,s_2)} : s_1 = -2, -2 + \frac{1}{t}, \cdots, 2;\; s_2 = -0.5, -0.5 + \frac{1}{t}, \cdots, 0.5 \right\},$$

with

$$R^t_s = \left[s_1 - \frac{1}{2t}, s_1 + \frac{1}{2t} \right] \times \left[s_2 - \frac{1}{2t}, s_2 + \frac{1}{2t} \right],$$

and the partition is refined by letting t be an arbitrarily large integer. With such a partition, a jump linear model of the tracking error dynamics is

$$x^{JL}(k+1) = A^{JL}_{s(k)} x^{JL}(k) + B^{JL}_{s(k)} u(k),$$

where

$$A^{JL}_s = \begin{bmatrix} -2as_1 & 1 \\ b & 0 \end{bmatrix}, B^{JL}_s = \begin{bmatrix} -2s_1 \\ 0 \end{bmatrix},$$

and $s(k) \in \mathbb{R}^2$ is the center of the cell $R^t_{s(k)}$ in which $\theta(k)$ falls, i.e., $\theta(k) \in R^t_{s(k)}$ with $\theta(k+1) = f(\theta(k), 0)$. The transition probabilities of s can be found by observing the trajectory of $f^k(\theta(0), 0)$ and using (15). While such observations will indicate that s is not a Markov chain, the results from this paper show that using the Markov approximation of the symbolic dynamics of s yields a controller that *still* stabilizes the nonlinear system.

This paper also shows that these two seemingly different approaches are closely related in the sense that, as the partition used to design the jump linear system is refined, i.e., as $t \to \infty$, the LDV and JL controllers converge. This convergence is illustrated in Figure 1 that shows the $(1,1)$ component of the solution Y_s to the coupled Riccati equations versus the cells of the partition of the horizontal (θ_1, θ_2) plane. Precisely, above each cell R^t_s of the partition, we draw a horizontal rectangle of the same (θ_1, θ_2) components as the cell and with vertical component the $(1,1)$ entry of the solution Y_s to the coupled Riccati equations. As the partition is refined, the figures indicate that the $(1,1)$ component of Y_s converges to the graph of a continuous function. This continuous function is the solution X_θ to the functional algebraic Riccati equation (4).

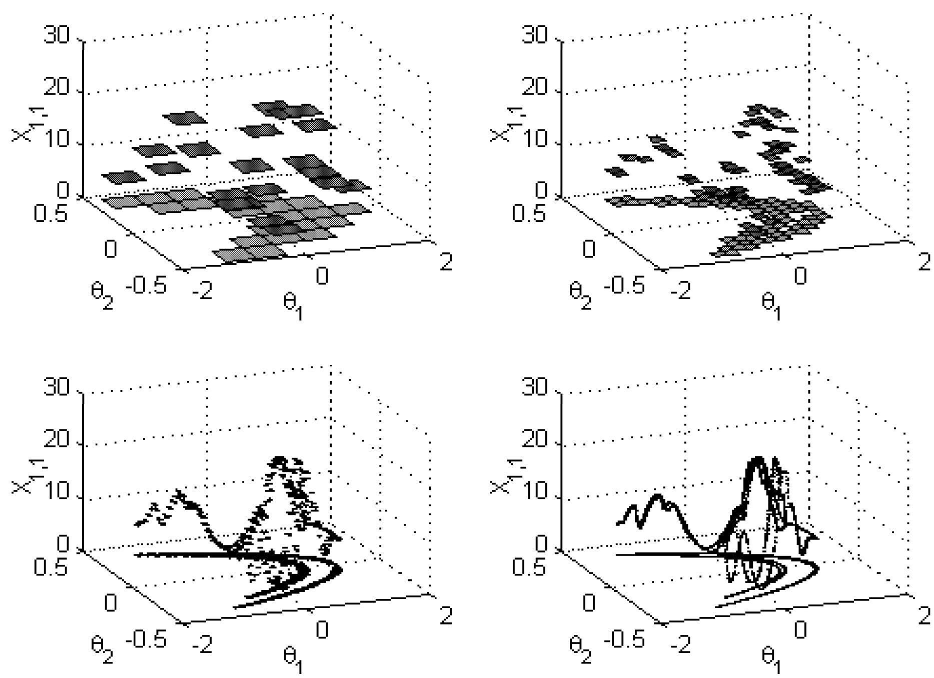

Figure 1: This figure shows the $(1,1)$ element of the Riccati solution of the optimal jump linear system as the partition is refined. The cost converges to a continuous function, which is the cost of the optimal LDV controller. For reference, the partition is shown in the $X_{1,1} = 0$ plane.

7 Concluding remark

When confronted with a nonlinear tracking problems, there are two seemingly distinct approaches–the linear dynamically varying (LDV) and the jump linear (JL) approaches. The JL tracking error model appears as a most natural approximation of the LDV tracking error model when the underlying nonlinear dynamics admits a symbolic dynamic representation on a Markov partition of the state space. The JL controller is an approximation of the LDV controller in the sense that, as the partition is refined, the JL controller converges to the LDV controller. It is noteworthy that these results *still hold* in the case where either the underlying nonlinear system does not admit a Markov partition or the Markov partition is unknown. This opens the possibility of solving the functional algebraic Riccati equation of LDV control via the coupled Riccati equations of JL control. However, a thorough numerical investigation of this possibility is left for future investigation.

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