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LDV Control over Compact Riemannian Manifolds

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***Abstract*— Linear Dynamically Varying (LDV) systems are a subset of Linear Parameter Varying (LPV) systems characterized by parameters that are dynamically modeled. An LDV system is, in most cases of practical interest, a family of linearized approximations of a nonlinear dynamical system indexed by the point around which the system is linearized. LDV systems emerge quite naturally in a generic trajectory tracking problem in which the tracking error is modeled as an LDV system and the tracking controller is tuned so as to minimize an LQ performance. This paper focuses on tracking a natural trajectory of a nonlinear dynamical system running over a Riemannian manifold. The associated LDV tracking error system runs over the tangent bundle and LQ minimization secures asymptotic tracking of the nonlinear trajectory. The LDV tracking controller is provided by the solution to a partial differential Riccati equation (PDRE), itself related to a linear partial differential Hamiltonian operator. The index of the latter operator reveals some ergodic properties of the reference flow.**

I. INTRODUCTION

A. Problem Definition

Given a differentiable n-D manifold Θ, a *dynamical system* over Θ [2] is defined, formally, by a vector field $f : \Theta \to T\Theta$. Here, we take $f \in \Gamma^\infty(\Theta, T\Theta)$, the space of *smooth* cross sections. The trajectories $\theta_{\theta_0}(t), \theta_0 \in \Theta, t \geq 0$ are the integral curves of the vector field f, that is,

$$\frac{d}{dt}\theta_{\theta_0}(t) = f(\theta_{\theta_0}(t)); \quad \theta_{\theta_0}(0) = \theta_0 \tag{1}$$

Next to the above nominal dynamics, we consider a controlled dynamical system

$$\frac{d}{dt}\varphi(t) = \tilde{f}(\varphi(t), u(t)); \quad u(t) \in \mathcal{U} \subseteq \mathbb{R}^p; \quad \varphi(0) = \varphi_0 \tag{2}$$

which is meant to be a perturbation of the nominal dynamics,

$$\tilde{f}(\theta, 0) = f(\theta); \quad f(\theta, \mathcal{U}) \subseteq T_\theta\Theta \tag{3}$$

In the above, $\mathcal{U}$ is a neighborhood of 0 in $\mathbb{R}^p$, and

$$\tilde{f} \in \Gamma^\infty_{\pi_1}(\Theta \times \mathcal{U}, T\Theta) \tag{4}$$

That is, $\tilde{f}$ is formally defined as a vector field along the projection on the first factor [13, Chap. III, Def. 1.4].

In dynamical systems, the control enters the dynamics in an affine fashion, so it will be assumed that $\tilde{f}$ takes the usual affine form

$$\tilde{f}(\varphi, u) = f(\varphi) + B(\varphi)u = f(\varphi) + \sum_{m=1}^{p} b_m(\varphi)u^m \tag{5}$$

where $b_m \in \Gamma^\infty(\Theta, T\Theta)$. By the same token, this secures invariance of Θ under the perturbation (see [20],[21]). The control objective is to find $u(t) \in \mathcal{U}$ such that, for θ_0 and φ_0 sufficiently close,

$$\lim_{t\to\infty} d(\varphi(t), \theta(t)) = 0$$

where $d(\cdot, \cdot)$ is an appropriately defined distance.

Because of space limitation, all proofs are deleted and will appear in the full paper.

B. Motivation

The original motivation for the above-defined control problem was tracking periodic and aperiodic trajectories embedded in a chaotic attractor [8], [9], [10], [7], with the *major* difficulty of proving θ_0-continuity of the controller even though the reference trajectory might not be time-uniformly continuous relative to θ_0. Probably the best illustrative example is tracking periodic and quasi-periodic orbits of a Trojan asteroid in libration around the Jupiter L4 point [3]. Recast in the abstract set-up of this paper, the control problem consists in picking up one of the Trojans by specifying its initial condition θ_0 and force the spacecraft (2) with initial condition φ_0 to go to a rendez-vous with it. Other applications include hovercraft control [4]. Yet another application along a slightly different line is cardiac dynamics, where the problem is *to avoid* periodic trajectories [16].

A more recent application, however, is the control of quantum systems described by the Liouville-von Neumann equation $\dot{\rho} = -\jmath[H_0 + \sum_{m=1}^p H_m u^m, \rho]$, where ρ is the density operator, the Hamiltonians H_i, $i = 0, 1, ..., p$, are Hermitian matrices, in a system of units where $\hbar = 1$. In the Pauli basis $\{e_k\}$ over $\mathbb{R}$ of the set of Hermitian matrices, the evolution takes the form of the *Bloch equation*, $\dot{\varphi} = A\varphi + \sum_{m=1}^p (B_m\varphi)\, u^m$, where φ is given in covariant coordinates as $\varphi_k = \text{Trace}(\rho e_k)$ with, from the Lie viewpoint, $A = \text{ad}(-\jmath H_0)$ and $B_m = \text{ad}(-\jmath H_m)$, $m = 1, ..., p$, while, from the differential geometry viewpoint, $A, B_m : T\Theta \to T\Theta$. In such quantum systems, tracking periodic and aperiodic trajectories has recently been considered, but more importantly, the latter has revealed the obstruction to the implementation of Lyapunov control schemes created by the nontriviality of Θ. Indeed, in the control of a pure qubit, φ evolves over the Bloch sphere Θ, a compact manifold. In closed systems, the evolution is isospectral, and Θ is a flag manifold [1], [23], again compact. If the evolution is subject to decoherence, then the problem is to control the system so that it evolves on a *Decoherence Free Manifold (DFM)*, which is a real analytic manifold of Hermitian matrices

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preserving some blocks of eigenvalues [15]. All such DFM's are compact.

C. Approach

Our approach to this task is an adaptation of the Linear Dynamically Varying (LDV) trajectory tracking techniques developed in [8], [9], [10]. In addition to providing a continuous-time version of [8] along with new proofs, the present paper addresses the specific issues arising from the fact that the dynamical system evolves over a smooth but nontrivial manifold Θ. The differential geometry setup reveals the issue of whether linearization should be done around the nominal or the controlled trajectory [11].

II. LDV SYSTEMS OVER RIEMANNIAN MANIFOLDS

A. Riemannian Metric and Connection

Define $\{e_i = \frac{\partial}{\partial \vartheta^i} : i = 1, ..., n\}$ to be the coordinate reference frame associated with the local coordinate system $\{\vartheta^i : i = 1, ..., n\}$. For $a \in T_\theta\Theta$, we write $a = a^i e_i$, where we make use of Einstein's convention.

First, we introduce a Riemannian metric $\langle a, b\rangle = g_{ij}a^i b^j$. Next, we introduce a covariant differentiation:

$$\nabla : \Gamma^\infty(\Theta, T\Theta) \times \Gamma^\infty(\Theta, T\Theta) \to \Gamma^\infty(\Theta, T\Theta),$$

defined by the extension of $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$ by linearity and the Leibniz rules to the covariant differentiation of any vector field along any other vector field. The notation $\nabla_{e_i} e_j$ denotes the *covariant derivative* of the vector field e_j along the vector field e_i. The Γ^k_{ij}'s are the Christoffel symbols of the connection.

The covariant derivative $\nabla_f v$ of a vector field v along the integral curve of some vector field, say f, will sometimes be rewritten as $\frac{D}{dt}v$, as a formalization of the intuitive notion of "the time derivative of a vector field v along the trajectories of f" (see [12, Proposition 2.2]).

Given a differentiable curve $c : [0, \ell] \to \Theta$ such that $c(s = \ell) = \varphi$ and $c(0) = \theta$, the tangent vector $h(c(t)) \in T_{c(t)}\Theta$ is said to be *parallel to itself* along c if $\nabla_{\frac{dc}{ds}} h = 0$. $h_0 \in T_{c(0)}\Theta$ is said to be coming from the *parallel transport* of $h_1 \in T_{c(1)}\Theta$ along c if $h_0 = h(c(0))$, where $h(c(t))$ is the solution to $\nabla_{\dot{c}} h = 0$ subject to $h(c(1)) = h_1$.

Let $\{e^j = \vartheta^j : j = 1, ..., n\}$ be the base dual to $\{e_i\}$, that is, $e^j(e_i) = \delta^j_i$. The covariant derivative of a cotangent vector along a tangent vector is obtained by extending $\nabla_{e_i} e^j = -\Gamma^j_{ki} e^k$ to any tangent vector and any cotangent vector by linearity and the Leibniz rule (see [5, p. 230] or [17, p. 106]).

B. Tracking Error

Consider the nominal motion (1) and the perturbed motion (2). To define the tracking error between $\varphi(t)$ and $\theta(t)$, assume that $\varphi(t)$ and $\theta(t)$ are so close that there exists a unique minimizing geodesic $\gamma_{\varphi(t)} : [0, \ell_{\varphi(t)}] \to \Theta$ joining them, viz., $\gamma_{\varphi(t)}(0) = \theta(t)$, $\gamma_{\varphi(t)}(\ell_{\varphi(t)}) = \varphi(t)$, where the parameterization is by the arclength. The tracking error, locally around $\varphi(t)$, is defined as

$$x_{\varphi(t)} = \ell_{\varphi(t)} \left. \frac{d\gamma_{\varphi(t)}}{ds} \right|_{s=\ell_{\varphi(t)}} \in T_{\varphi(t)}\Theta$$

Now assume that the nominal trajectory is fixed, but that the controlled system could potentially start anywhere. Assume that the control u is a smooth scalar field so that $\tilde{f}$ is a smooth vector field over Θ. The latter in turn defines a flow $\tilde{\phi}_t$, which encompasses the nominal flow. The same can be said for the case of an open-loop control $u(t)$ provided the controlled trajectories do not cross. This setup allows us to extend $x_{\varphi(t)}$, initially defined over a specific controlled trajectory, to a vector field $x_{\bar{\varphi}}$. Indeed, define $\bar{\varphi} = \tilde{\phi}_t \bar{\varphi}_0$. Let $\gamma_{\bar{\varphi}} : [0, \ell_{\bar{\varphi}}] \to \Theta$ be the geodesic joining $\theta(t)$ to $\bar{\varphi}$, parameterized by arc length. Then

$$x_{\bar{\varphi}} = \ell_{\bar{\varphi}} \left. \frac{d\gamma_{\bar{\varphi}}}{ds} \right|_{s=\ell_{\bar{\varphi}}} \in T_{\bar{\varphi}}\Theta$$

The need for this extension will become clearer soon.

C. Tracking Error Dynamics

The rate of change of the tracking error is defined as the covariant derivative of the tracking error x along the field $\tilde{f}$. However, $\tilde{f}$ incorporates the control u, which has both an open-loop and a closed-loop interpretation. In the open-loop interpretation, $u(t)$ is a time-varying parameter independent of θ and x. In the closed-loop interpretation, u has the form of a linear feedback in x. It is therefore important to validate the error dynamics for both interpretations.

Before doing so, we need two lemmas:

Lemma 1: If v and w are two vector fields defined over a differentiable manifold endowed with a symmetric connection,

$$\nabla_v w = \nabla_w v + [v, w]$$

where $[v, w]$ is the Lie bracket of v and w defined as $vw - wv$, for the partial differential operator interpretation of the vector fields. ■

Denote by $o(||x||)$ a function such that $o(||x||)/||x|| \to 0$ as $||x|| \to 0$ (see [6, Sec. 1.8]).

Lemma 2: With the vector fields x, f, and $\tilde{f}$ defined as above, $||[\tilde{f}, x](\varphi_0)|| = o(||x(\varphi_0)||) = o(d(\theta_0, \varphi_0))$. Consequently, $||[f, x](\varphi_0)|| = o(||x(\varphi_0)||) = o(d(\theta_0, \varphi_0))$. ■

Remark 1. It is because the field x is derived from the field f that the bracket behaves as $||[f, x]|| = o(||x||)$, which *appears* to contradict the fact that $[f, x]$ is linear in x. Also observe that, if f is a geodesic field, then $[f, x] = 0$.

1) Open-Loop Form of Control: In its open-loop formulation, $u(t)$ is a function of the time generated by its own dynamics, say $\dot{u} = h(u)$, where $h \in C^1$ and $h(0) = 0$, the latter to enforce the fact that u is small. Using the invariance relation (3), the augmented dynamics over the product manifold $\Theta \times \mathcal{U}$ is defined via the augmented vector field:

$$\begin{array}{rcl}\Theta\times\mathcal{U} & \to & T\Theta\times(\mathcal{U}\times\mathbb{R}^p)\\ (\theta,u) & \mapsto & (\tilde{f}(\theta,u),h(u))\end{array} \tag{6}$$

Then, for the augmented dynamics, we have

$$\begin{array}{rl}\frac{D}{dt}\begin{pmatrix}x\\u\end{pmatrix}=\nabla_{(\tilde{f},h)}\begin{pmatrix}x\\u\end{pmatrix} & \\ = & \nabla_{(f+Bu,h)}\begin{pmatrix}x\\u\end{pmatrix},\ \text{(see (5) for definition of } B)\\ = & \nabla_f\begin{pmatrix}x\\u\end{pmatrix}+\nabla_{(Bu,h)}\begin{pmatrix}x\\u\end{pmatrix}\\ = & \nabla_f\begin{pmatrix}x\\u\end{pmatrix}+\nabla_{(x,u)}\begin{pmatrix}Bu\\h\end{pmatrix}+\left[\begin{pmatrix}Bu\\h\end{pmatrix},\begin{pmatrix}x\\u\end{pmatrix}\right]\\ \approx & \nabla_f\begin{pmatrix}x\\u\end{pmatrix}+\nabla_{(x,u)}\begin{pmatrix}Bu\\h\end{pmatrix}\end{array}$$

($\approx$ denotes "up to first order.") Taking the top equation yields

$$\begin{array}{rcl}\frac{D}{dt}x & = & \nabla_f x+\sum_{m=1}^{p}\left(\frac{\partial b_m^i}{\partial\vartheta^j}+\Gamma_{jk}^i b_m^k\right)x^j e_i u^m+Bu\\ & \approx & \nabla_f x+Bu=\nabla_x f+[f,x]+Bu\\ & \approx & \nabla_x f+Bu\end{array}$$

2) Feedback Form of Control: In this LDV setup, we are aiming at an invariant description of the feedback control in the form

$$u^m=\langle k_m,x\rangle,\quad m=1,...,p \tag{7}$$

where k_m is the control gain. Hence $\tilde{f}$ is a vector field and we have

$$\begin{array}{rcl}\frac{D}{dt}x & = & \nabla_{\tilde{f}}x=\nabla_{f+Bu}x=\nabla_f x+\nabla_{Bu}x\\ & = & \nabla_x f+[f,x]+\nabla_x Bu+[x,Bu]\\ & \approx & \nabla_x f+\nabla_x Bu\\ & = & \nabla_x f+\sum_{m=1}^{p}((\nabla_x b_m)u^m+(xu^m)b_m)\\ & \approx & \nabla_x f+\sum_{m=1}^{p}(xu^m)b_m\end{array}$$

In the above, xu^m denotes the directional derivative [12, Sec. 0.5] of u^m along x. Since the directional derivative $x\langle k_m,x\rangle$ does not depend on the parameterization [12, p. 25], choose a coordinate frame such that $x=\xi e_1$, in which case

$$\begin{array}{rcl}x\langle k_m,x\rangle & = & x\left(k_m^1\xi e_1\right)=\xi\left.\frac{\partial}{\partial\xi}k_m^1(\xi)\xi e_1(\xi)\right|_{\xi=0}\\ & = & k_m^1(0)\xi e_1(0)=u^m\end{array}$$

Therefore,

$$\frac{D}{dt}x=\nabla_x f+Bu \tag{8}$$

Remark 2. At this stage, the need for the rather clumsy extension of the tracking error to a neighborhood of the controlled trajectory can be understood. Clearly the preceding analysis relies crucially on the Lie bracket $[f,x]$ and for this bracket to exist x needs to be a vector field. The covariant derivative $\nabla_f x$ on the other hand does not require x to be a vector field (see, e.g., [5, Problem 5.8.2], [18, p. 124], [12, Remark 2.3]).

D. LDV Dynamics along Controlled Trajectory

For both the open-loop formulation (6) and the closed-loop formulation (7) of the control, the error dynamics is given by (8). It remains to evaluate $\nabla_x f$, which is easily done:

$$\nabla_x f=e_i\left(\frac{\partial f^i}{\partial\vartheta^j}+\Gamma_{jk}^i f^k\right)x^j \tag{9}$$

Therefore, the linearized dynamics for x is

$$\frac{D}{dt}x(t)=A_{\varphi(t)}x(t)+B_{\varphi(t)}u(t) \tag{10}$$

where

$$A=A_j^i e_i\otimes e^j,\quad A_j^i=\frac{\partial f^i}{\partial\vartheta^j}+\Gamma_{jk}^i f^k \tag{11}$$

and B is given by (5). (10), together with (2)-(5) is called a *Linear Dynamically Varying (LDV)* system, along the controlled trajectory.

E. LDV Dynamics along Nominal Trajectory

Although $T_{\varphi(t)}\Theta$ is the nominal tangent space in which the controller operates, unfortunately, minimizing such a cost criterion as $\int_0^\infty\left(||C_{\varphi(t)}x(t)||^2_{T_{\varphi(t)}\Theta}+\langle u(t),Ru(t)\rangle_{T\mathcal{U}}\right)dt$ subject to (10) and (2) yields quite a complicated nonlinear controller (see Remark 8). To design a linear controller, we have to move $\nabla_{\tilde{f}}x=A_\varphi x+B_\varphi u$ from $T_{\varphi(t)}\Theta$ to $T_{\theta(t)}\Theta$ by parallel translation along γ and relinearize around the nominal trajectory θ. To prove the validity of this operation in this first order analysis, we have to show that it entails an error of the order of $o(||x||)$.

Clearly, under parallel translation along $\gamma_{\varphi(t)}$, x remains tangent to the geodesic and, more importantly, the crucial tracking error magnitude $||x||$ remains unchanged. Let v be a generic notation for any of the vector fields x, $\nabla_{\tilde{f}}x$, Bu. Choose $\exp^{-1}_{T_\varphi\Theta}$ as coordinate chart and let ϑ^k be the coordinate functions. Write $v=v^k e_k$, where $e_k=\frac{\partial}{\partial\vartheta^k}$. Then, in those coordinates, the parallel translation of v along the curve γ is a solution to the differential equation (see [12, pp. 52-53])

$$\frac{dv^k}{ds}=-\Gamma_{ij}^k v^j\frac{d\vartheta^i(\gamma)}{ds}$$

Since the translation is along a geodesic, $\frac{d\vartheta^i(\gamma)}{ds}=0$, so that v^k remains constant in the local coordinate system during the translation. Therefore, during the parallel translation, the components of x, $\nabla_{\tilde{f}}x$, and Bu in the local coordinate system remain unchanged. By a slight generalization of the above argument, namely by working out $\nabla_{\dot{\gamma}}\left(A_j^i e_i\otimes e^j\right)=0$ (see [18, Prop. 2.11]), the components of the tensor field A during the parallel transport are easily seen to satisfy

$$\frac{dA_j^i}{ds}=-A_j^m\Gamma_{km}^i\frac{d\vartheta^k(\gamma)}{ds}+A_m^i\Gamma_{jk}^m\frac{d\vartheta^k(\gamma)}{ds}$$

so that the components of A remain unchanged during the parallel translation. With a slight abuse of notation, let the result of this parallel translation be written $\left(\nabla_{\tilde{f}}x\right)_\varphi=$

$A_\varphi x + B_\varphi u$, linearized around φ, but transported to $T_\theta\Theta$. In $T_\theta\Theta$, rewrite the preceding as

$$\nabla_f x + (\nabla_{\tilde{f}} x - \nabla_f x) = A_\theta x + (A_\varphi - A_\theta)x + B_\theta u + (B_\varphi - B_\theta)u \tag{12}$$

and let us show that the terms between parentheses are $o(||x||)$. Clearly, in the above local coordinates, $(A_\varphi - A_\theta)^i_j$, $(B_\varphi - B_\theta)^i_j$ are of the order of $||x||$. It remains to look at $(\nabla_{\tilde{f}} x)_\varphi - (\nabla_f x)_\theta$. In the same coordinate system as above, we have

$$\begin{aligned}
&(\nabla_{\tilde{f}} x)_\varphi - (\nabla_f x)_\theta \\
&= \tilde{f}^i(\varphi)\left(\frac{\partial x^j}{\partial \vartheta^i} + \Gamma^j_{ik}x^k\right)_\varphi e_j - f^i(\theta)\left(\frac{\partial x^j}{\partial \vartheta^i} + \Gamma^j_{ik}x^k\right)_\theta e_j \\
&= \left(\tilde{f}^i(\varphi)\left.\frac{\partial x^j}{\partial \vartheta^i}\right|_\varphi - f^i(\theta)\left.\frac{\partial x^j}{\partial \vartheta^i}\right|_\theta\right) e_j \\
&\quad + \left(\tilde{f}^i(\varphi)\Gamma^j_{ik}(\varphi) - f^i(\theta)\Gamma^j_{ik}(\theta)\right) x^k e_j
\end{aligned}$$

where, in the second equality, we have used the fact that, during the parallel transport from $\varphi(t)$ to $\theta(t)$, x remains unchanged. Clearly, the second term of the second equation is $o(||x||)$. Regarding the first term, observe that $\tilde{f}^i \frac{\partial x^j}{\partial \vartheta^i}$ is the directional derivative of x^j relative to $\tilde{f}$; as such, if $\theta(t)$ and $\varphi(t)$ are local integral curves of f, $\tilde{f}$, respectively, it follows that

$$\left(\tilde{f}^i \frac{\partial x^j}{\partial \vartheta^i}\right)_\varphi - \left(f^i \frac{\partial x^j}{\partial \vartheta^i}\right)_\theta = \frac{dx^j(\varphi(t))}{dt} - \frac{dx^j(\theta(t))}{dt}$$

But, by definition, x is unchanged from $\varphi(t)$ to $\theta(t)$, and the above vanishes. Therefore, $(\nabla_{\tilde{f}} x)_\varphi - (\nabla_f x)_\theta$ is $o(||x||)$. Therefore, from (12), it follows that, up to a $o(||x||)$ term, the linearized dynamics becomes

$$\nabla_f x = A_{\theta(t)} x + B_{\theta(t)} u \tag{13}$$

where A, B are still given by (11,5), except that the partial derivatives are evaluated at $\theta(t)$. The above is called the *Linear Dynamically Varying* system, along the nominal trajectory.

Remark 3. Observe that x is not a vector field along the nominal trajectory, but is rather the parallel translation of a vector field along the controlled trajectory to the nominal trajectory. Note that for the covariant derivative to exist, there is no need for a vector field; a tangent vector along the nominal trajectory is enough.

Remark 4. There is a possibility to derive (13) in a more direct fashion. This is done by defining the tracking error around the nominal trajectory as the tangent to the geodesic $\gamma_{\varphi(t)}$ at $\theta(t)$ (rather than at $\varphi(t)$). The problem is that the $o(||x||)$ analysis requires x to be a vector field so that x, as initially defined, needs to be extended, a bit arbitrarily, to a neighborhood of the nominal trajectory. This can be justified in situations where there is only one controlled trajectory but many nominal trajectories to choose. For example, a spacecraft entering the Trojan asteroid pack around L4 would have to choose which asteroid to track. After this extension, $\frac{D}{dt}x = A_{\theta(t)}x$ would be rather easily derived, but the control would still have to be "imported" from $T_{\varphi(t)}\Theta$.

The solution to the free dynamics defines the state transition operator:

$$\begin{aligned}
\Phi_{\theta_0}(t,0) : T_{\theta_0}\Theta &\to T_{\theta(t)}\Theta \\
x_0 &\mapsto x(t)
\end{aligned}$$

equivalently defined as the solution to

$$\begin{aligned}
\frac{D}{dt}\Phi_{\theta_0}(t,0) &= A_{\theta(t)}\Phi_{\theta_0}(t,0) \\
\Phi_{\theta_0}(0,0) &= I_{T_{\theta_0}\Theta}
\end{aligned}$$

Observe that $\Phi_{\theta_0}(t,0)$ is *not* in general $d_{\theta_0}\phi_t$, unless the connection is "flat."

Remark 5. Observe that $\nabla_x f$ *does not* need the extension of x originally defined everywhere on the nominal trajectory; however, $[f,x]$ does need the extension of x to a vector field defined in the tubular neighborhood of the nominal trajectory.

The computationally attractive matrix representations for A_θ, B_θ are guaranteed to exist, *locally*. To obtain *global* matrix representations for A_θ, B_θ, it is necessary to have a coordinate frame $\{e_{1\theta}, ..., e_{n\theta}\}$ in the tangent space $T_\theta\Theta$, smoothly depending on θ. This is the issue of the *parallelizability* of the manifold Θ. If Θ is not parallelizable, it may have a parallelizable covering space; in this case, after lifting the problem to the covering space, we have global matrix representations.

III. FINITE HORIZON PROBLEM

The tracking objective will be achieved by minimizing, along the nominal trajectory, the integral of $||C_{\theta(t)}x(t)||^2_{T_{\theta(t)}\Theta} + \langle u(t), Ru(t)\rangle_{T\mathcal{U}}$, where $C : T\Theta \to T\Theta$ is a bundle map or equivalently a $(1,1)$ tensor field. The positive semidefinite quadratic form $||Cx||^2$ can easily be rewritten as $x^k(C^i_k g_{ij} C^j_l x^l)$. Define $(Qx)_k = C^i_k g_{ij} C^j_l x^l$, where $Q = Q_{kl} e^k \otimes e^l \in T^*\Theta \otimes T^*\Theta$, $Q_{kl} = C^i_k g_{ij} C^j_l$, is a bilinear form, that is, a $(0,2)$ tensor field or equivalently a bundle map $T\Theta \to T^*\Theta$ (see [5, Sec. 2.20, 2.21]). Q will be rewritten more compactly as C^*GC, where $G : T\Theta \to T^*\Theta$ is the fundamental tensor and $C^* : T^*\Theta \to T^*\Theta$ the dual of C. With this convention, $||Cx||^2 = \langle x, Qx\rangle$, where the latter is defined as $\sum_i x^i(Qx)_i$, that is, the *bilinear pairing* [17, p. 105] between the tangent vector x and the cotangent vector Qx. In the sequel, the distinction between Riemannian metric and bilinear pairing will only be made by the nature of the second argument, tangent versus cotangent vector, resp., with the latter written by a Greek letter, if possible (see [5, p. 131] for the same convention). The same convention applies to $\langle x_0, X_{\theta_0}x_0\rangle$, where x_0, the initial condition, is a tangent vector and X_{θ_0}, the "Riccati solution," is a $(0,2)$ tensor field.

Assumptions A

1) Θ is a differentiable manifold with a symmetric connection ∇.
2) A and B are continuous.
3) Q is a positive semidefinite quadratic form.
4) R is a positive definite quadratic form.

Theorem 1: Let Assumptions A hold. Then there exists a bundle map

$$X(\theta_0, 0, T) : T_{\theta_0}\Theta \to T^*_{\theta_0}\Theta$$

differentiable relative to θ_0 along the flow, continuous across the flow, and such that

$$\langle x_0, X(\theta_0, 0, T)x_0\rangle = \inf_u \int_0^T \left(\langle x(\tau), Qx(\tau)\rangle + \langle u(\tau), Ru(\tau)\rangle\right) d\tau$$

Furthermore, X satisfies the Riccati differential equation, in coordinate independent format,

$$\begin{aligned} &\frac{D}{dt}X(\theta_0,t,T) + A^*_{\theta(t)}X(\theta_0,t,T) + X(\theta_0,t,T)A_{\theta(t)} \\ &\quad +Q_{\theta(t)} - X(\theta_0,t,T)B_{\theta(t)}S_{\theta(t)}B^*_{\theta(t)}X(\theta_0,t,T) = 0 \\ &X(\theta_0,T,T) = 0 \end{aligned} \tag{14}$$

or equivalently, in terms of coordinates or matrix representation relative to the basis $\{e^i \otimes e^j : i,j = 1,...,n\}$,

$$\begin{aligned} &\frac{d}{dt}X_{ij}(\theta_0,t,T) + F^k_i X_{kj}(\theta_0,t,T) + X_{ik}(\theta_0,t,T)F^k_j \\ &\quad +Q_{ij} - X_{ik}(\theta_0,t,T)B^k_l S^{lm}B^q_m X_{qj}(\theta_0,t,T) = 0 \\ &X(\theta_0,T,T) = 0 \end{aligned} \tag{15}$$

where $S : T^*\mathcal{U} \to T\mathcal{U}$ is the inverse mapping of R and $F^i_j := \frac{\partial f^i}{\partial \vartheta^j}$. ■

It should be *warned* that the apparent simplification afforded by the cancellation of the Γ's in (14) leading to (15) has made the latter *not invariantly defined*. Indeed, it is easily verified that the format of (15) is changed under a coordinate neighborhood transformation $\vartheta = u_2 \circ u_1^{-1}(\vartheta')$. In other words, $f^i \frac{\partial X_{kl}}{\partial \vartheta^i}$ are *not* the components of a tensor. This problem disappears if Θ is an affine manifold, that is, the coordinate maps u_α can be chosen so that the $u_\alpha \circ u_\beta^{-1}$'s are affine transformations.

IV. INFINITE HORIZON PROBLEM

A. Stability, Stabilizability, Detectability

Definition 1: The LDV system $\frac{D}{dt}x(t) = A_{\theta(t)}x(t)$ is said to be *exponentially stable* (along the flow of f) if there exist functions $\alpha(\theta_0) \in (0,\infty)$ and $\beta(\theta_0) \geq 0$ such that, for every $x_0 \in T_{\theta_0}\Theta$, $||x(t)|| \leq \beta(\theta_0)e^{-\alpha(\theta_0)t}||x(0)||$ The system is said to be *uniformly exponentially stable* iff there are numbers $\alpha \in (0,\infty)$, $\beta \geq 0$ such that $||x(t)|| \leq \beta e^{-\alpha t}||x(0)||$ Finally, the system is said to be *asymptotically stable* iff $\lim_{t\to\infty} ||x(t)|| = 0$

Theorem 2: If Θ is compact and the LDV system $\frac{D}{dt}x(t) = A_{\theta(t)}x(t)$ continuous, then asymptotic stability, exponential stability, and uniformly exponential stability are equivalent. ■

Definition 2: The LDV system $\frac{D}{dt}x = A_\theta x + B_\theta u$ is said to be *stabilizable* iff there exists a bundle function $K : T\Theta \to \mathbb{R}^p$ such that $\frac{D}{dt}x = (A_\theta + B_\theta K_\theta)x$ is stable and the resulting state transition operator $\Phi_{\theta_0} : T_{\theta_0}\Theta \ni x_0 \mapsto x(\cdot) \in T\Theta$ is uniformly bounded in the sense that $\int_0^\infty \langle \Phi_{\theta_0}x_0(t), \Phi_{\theta_0}x_0(t)\rangle dt < M(x_0)$, for some $M(x_0)$ independent of θ_0.

Remark 6. The bundle function K need not be continuous; to be specific, $K_\theta : T_\theta\Theta \to \mathbb{R}^p$ need not continuously depend on θ.

Clearly, for the minimization of the cost to yield a stabilizing controller, some detectability conditions are needed.

Definition 3: The LDV system $\frac{D}{dt}x(t) = A_\theta(t)x(t)$, $z(t) = C_{\theta(t)}x(t)$ is said to be detectable if there exists a bundle function $L : T\Theta \to T\Theta$ such that $\frac{D}{dt}x = (A + LC)x$ is stable.

Remark 7. Again, the bundle function $L : T\Theta \to T\Theta$ need not be continuous in θ

dual system $\frac{D}{dt}\xi = A^*\xi + C^*\mu$.

Lemma 3: If the LDV system $\frac{D}{dt}x = Ax$, $z = Cx$ is detectable, then $z \to 0$ implies that $x \to 0$. ■

Observe that the conditions of stability, stabilizability, and detectability are the properties of the various trajectories, while the uniform versions are the properties of the *flow*.

Theorem 3: If there exists a positive definite quadratic form, that is, a $(0,2)$-tensor field P_θ such that

$$\nabla_f P + A^*P + PA = -C^*GC$$

where (A,C) is detectable, then A is stable. ■

B. Main Result

Assumptions B

1) (A,B) is stabilizable.
2) (A,C) is detectable.

Theorem 4: Let Assumptions A and B hold. Then there exists a bundle map

$$X : T\Theta \to T^*\Theta$$

differentiable relative to θ_0 along the flow, continuous across the flow, and such that

$$\langle x_0, X_{\theta_0}x_0\rangle = \inf_u \int_0^\infty \left(\langle x(\tau), Q_{\theta(\tau)}x(\tau)\rangle + \langle u(\tau), Ru(\tau)\rangle\right) d\tau$$

Furthermore, X satisfies the partial differential Riccati equation (PDRE), in coordinate independent format,

$$\nabla_f X_\theta + A^*_\theta X_\theta + X_\theta A_\theta + Q_\theta - X_\theta B_\theta R^{-1}B^*_\theta X_\theta = 0 \tag{16}$$

or equivalently, in terms of coordinates or matrix representation relative to the dual basis $\{e^i\}$,

$$\sum_k \frac{\partial X_{ij}}{\partial \vartheta^k} f^k(\theta) + F^k_i X_{kj} + X_{ik}F^k_j + Q_{ij} - X_{ik}B^k_l S^{lm}B^q_m X_{qj} = 0 \tag{17}$$

and such that $A_\theta - B_\theta R^{-1}B^*_\theta X_\theta : T\Theta \to T\Theta$ is asymptotically stable along the flow. ■

It is important to observe that the PDRE for X involves a *very particular* partial differential operator, the covariant derivative of X *along the flow of* f. However, *no claims* have yet been made about differentiability of X *across trajectories*, that is, differentiability along directions not aligned with the flow of f. This issue is made complicated by the fact that systems running over compact sets exhibit sensitive dependence on initial conditions; that is, no matter how small $d(\theta_0, \theta_0')$ is, the trajectories flowing out of θ_0, θ_0' could be quite different, and as such it is counterintuitive that X_θ is continuous across the flow. To secure differentiability of X_θ, we need to strengthen the conditions of the previous theorem:

Theorem 5: If the conditions of Theorem 4 on A and B are strengthened to continuity and continuous differentiability of A, B, then $X \in W^{1,2}(\Theta)$, where $W^{1,2}(\Theta)$ denotes the Sobolev space of functions square integrable along with their first order derivatives over Θ. ■

Theorem 6: The PDRE has a unique asymptotically stabilizing, differentiable solution. This solution, X, is the maximum of all solutions, in the sense that for any other solution Y, we have $X_\theta - Y_\theta \geq 0, \forall \theta$. ■

Remark 8. In the infimization of $\int_0^\infty \left(\langle x, Qx \rangle_{T_{\varphi(t)}\Theta} + ||u||^2_{\mathcal{U}} \right) dt$ subject to (10), the constraint is no longer linear, because the control u enters implicitly in φ. Therefore, the controller will no longer be linear.

C. Stability of Nonlinear System under LDV Controller

Now that we know that the LDV controller stabilizes the linearized error dynamics, it remains to determine whether the same LDV controller also stabilizes the full nonlinear dynamics.

Theorem 7: Under Assumptions A and B, and under the strengthened condition that $Q > 0$, there exists a neighborhood of θ_0 in $T_{\theta_0}\Theta$, $\mathcal{O}_{\theta_0}$, such that $\forall \varphi_0 \in \exp_{\theta_0}(\mathcal{O}_{\theta_0})$ and with the LDV controller in place, we have $d(\theta(t), \varphi(t)) \to 0$ as $t \to \infty$. ■

V. HAMILTONIAN OPERATOR

Not surprisingly, the PDRE involves the partial differential operator P (details are available in [14]):

$$\begin{pmatrix} A - \nabla_f & -BR^{-1}B^* \\ -Q & -A^* - \nabla_f \end{pmatrix} : \Gamma^\infty(\Theta, TT^*\Theta) \to \Gamma^\infty(\Theta, TT^*\Theta)$$

Theorem 8: Let Θ be a compact, smooth, oriented Riemannian manifold with a natural volume form $\nu_\theta = \sqrt{|g|} d\vartheta^1 \wedge d\vartheta^2 \wedge \cdots \wedge d\vartheta^n$ and a connection compatible with the metric. Assume there exists a weighted volume form $\omega_\theta = a\nu_\theta$ such that ω is preserved along the flow f, that is, $L_f \omega = 0$ (see [19, Chapter 1, Section 3] or [17] for details). Then, for the measure $\mu_\omega(d\theta)$ induced by the weighted volume form ω, the operator P is Hamiltonian, that is, skew self-adjoint for the symplectic form. ■

It turns out that, in the case where f is an Axiom A flow [22, II.5.1], the operator P restricted to the nonwandering set Ω is Fredholm with bounds on its analytical index in terms of the spectral decomposition of the flow [22, II.5.2].

VI. CONCLUSION AND FURTHER PROSPECTS

LDV provides a general, *local* method to design tracking controllers, at the expense of requiring nontrivial stabilizability conditions on (A, B). Since, by Eq. (11), the pair (A, B) depends on the connection, which is arbitrarily imposed up to compatibility with a metric from the moduli space of Riemannian metrics over Θ, the challenge would be to figure out with what connection the design would work best.

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