Geometric topology puzzle in networking:

Core versus anti-core of classical versus quantum networks

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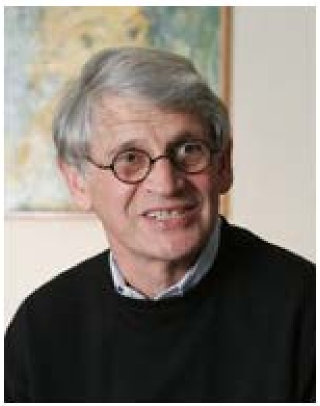
Paradoxical statements

- Boris Rozovskii (USC)
- S. S. Sritharan (DARPA)



“If the Internet has worked it is because there was no mathematics in it!”

“The time is ripe for mathematicians, statisticians, and control persons to have a serious look at Internet security!”

But, a “dirty little secret” remained unexplained:

Why are *a very few* routers *very badly* congested?

What mathematics to expect?

- In classical networks:
  - Routing-driven
    - Dijkstra
    - Single-path flow
    - Congestion core
  - Coarse metric geometry
    - Hop-distance
    - Gromov boundary ∂_∞
    - Geometric topology
    - ???
      - ???
  - ???
    - ???
    - ???
    - ???

- In quantum networks:
  - Physics-driven
    - ???
    - Feynman multi-path integral
    - Anti-core
  - Coarse metric geometry
    - $d(i,j) = -\log p_{\max}(i,j)$
    - Gromov boundary ∂_∞
    - ???
    - Projective geometric
      - **Global phase**
  - Number theory
    - Simultaneous Diophantine approximation
    - LLL-algorithm
    - Riemann zeta function

Plan of action

- Classical networks
  - Coarse geometry
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  - Load balancing
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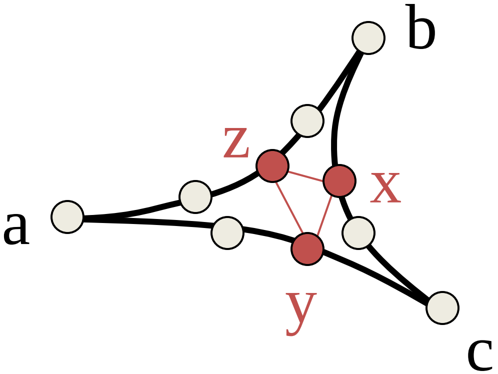
Gromov-hyperbolic geodesic spaces

A geodesic space (G, d) is ***Gromov hyperbolic*** iff every triangle has an inscribed triangle with its perimeter not exceeding a precise bound:

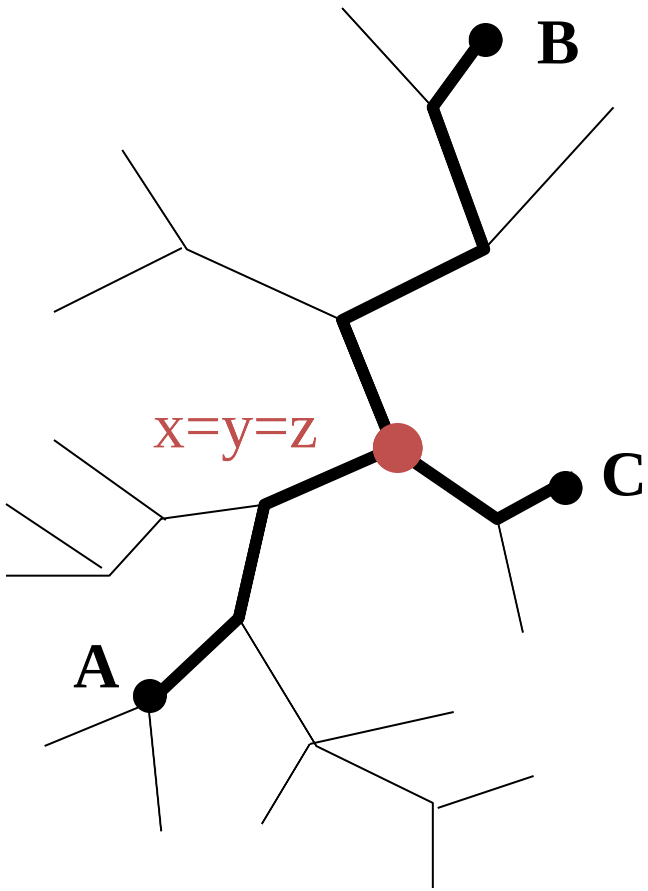
$$\delta(G) := \sup_{a,b,c \in G} \overbrace{\inf \left\{ d(x,y) + d(y,z) + d(z,x) : \begin{array}{r} x \in [b,c] \\ y \in [c,a] \\ z \in [a,b] \end{array} \right\}}^{\delta(\Delta abc)} < \infty$$



If $G = (V, E)$ is a graph, then

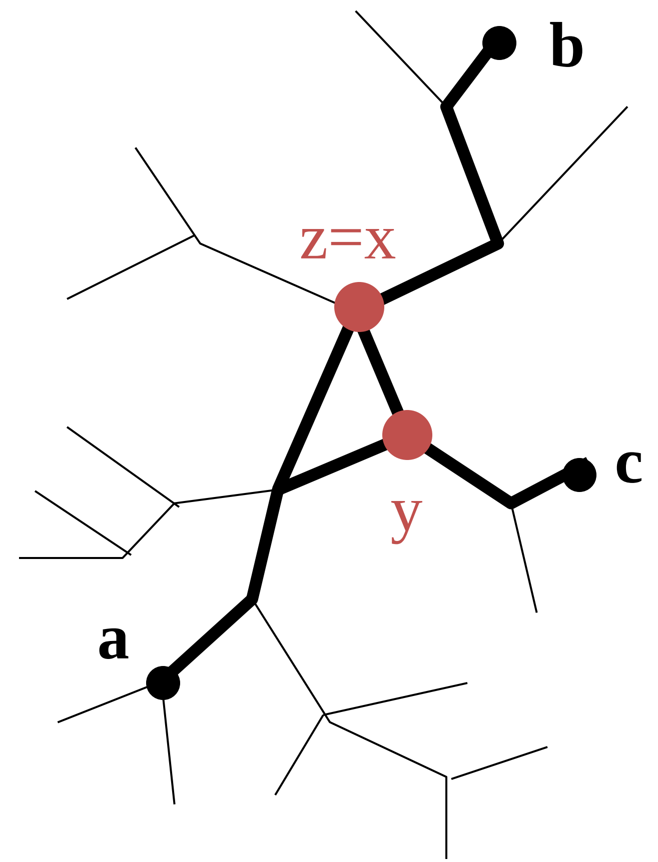
$$\arg\inf \left\{ d(x,y) + d(y,z) + d(z,x) : \begin{array}{r} x \in [b,c] \\ y \in [c,a] \\ z \in [a,b] \end{array} \right\} \in V$$

A "fattened tree" is Gromov hyperbolic, but not conversely!



A tree is $(\delta = 0)$-fat.

Δabc are geodesic triangles



$(\delta = 2)$

Connection between hyperbolic spaces in Riemannian and Gromov sense

<u>Theorem (Bonk-Schramm):</u> Let (G, d_G) be a Gromov hyperbolic geodesic metric space with bounded growth at some scale. Then there exist an integer n, a convex subset $D \subseteq \mathbb{H}^n$, constants λ, k, and a map $f: G \to D$ such that

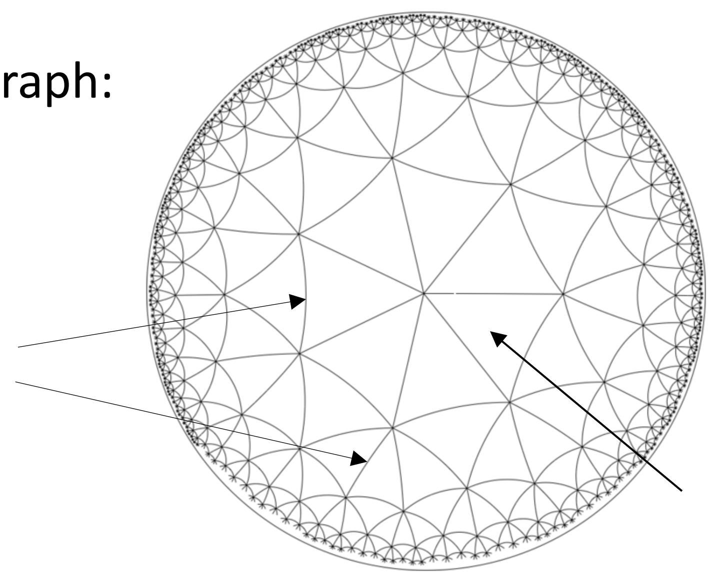
$$\left|\lambda d_G(u,v) - d_D(f(u), f(v))\right| \le k, \ \forall u,v \in G$$

$$\sup_{x \in D} d_D(x, f(G)) \le k, \ \forall x \in D$$

Gromov hyperbolic graph:

Poincaré disk:

$$(G, d_G)$$



$$\left(M_{-1}^2, \quad d_D(x,y) = \tanh^{-1}\left|\frac{x-y}{1-x\bar{y}}\right| \right)$$

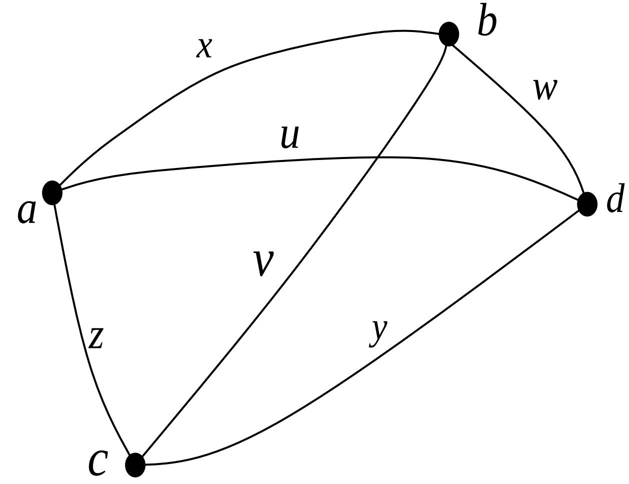
4-point computational implementation of Gromov δ

Given a complete quadrilateral $abcd$, order the sum of the lengths of opposite diagonals as

$$L = u + v \geq M = x + y \geq S = z + w$$

Define

$$\delta_4(abcd) = \frac{L - M}{2}$$



Then the geodesic space (G, d) is ***Gromov hyperbolic*** iff

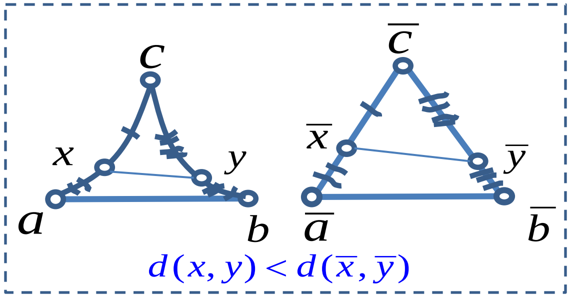
$$\sup_{abcd \subseteq G} \delta_4(abcd) < \infty$$

Scaling of Gromov δ_4

The Gromov coarse geometry makes sense only for *infinite spaces*, while real-life graphs, no matter how large, are finite.

We compute the upper bound of $\frac{\delta_4}{D}$ in standard spaces,

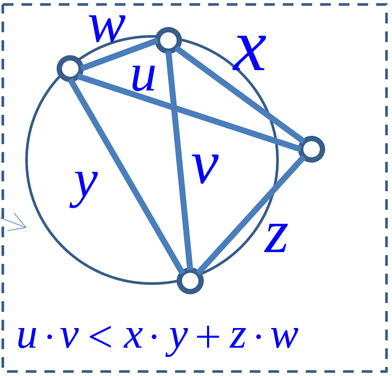
- hyperbolic,
- Cartan-Alexandrov-Toponogov CAT(0)
- Ptolemaic



for various scalings *D:*

$$\sup_{\substack{a,b,c,d\in M \\ A\xi\leq 0, c(\xi)\leq 0}} \frac{\delta_4(a,b,c,d)}{D(a,b,c,d)}, \quad \text{where} \quad D(a,b,c,d) = \begin{cases} L \\ L+M+S \\ \text{diam} \end{cases}$$

	hyperbolic	CAT(0)	Ptolemaic	Spherical
L	0.1307	0.1464	0.1667	0.25
$L+M+S$	0.0572	0.0607	0.0714	0.125
diam	0.2788	0.2929	0.2929	0.5



By Tarski-Seidenberg decision: $$\left(\frac{\delta_4(abcd)}{L+M+S}\right)_{\text{CAT}(0)} < \frac{\sqrt{2}-1}{2\sqrt{2}+4}$$

E. Jonckheere, P. Lohsoonthorn, and F. Ariaei, ``Scaled Gromov four-point condition for network graph curvature computation,“ *Internet Mathematics,* volume 7, number 3, pp. 137-177, 2011.

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Traffic load and density of traffic load in *U*

The TCP-IP protocol sends traffic from a source ***s*** to a destination ***d*** along a geodesic.

The demand $s \to d$ is uniformly distributed over all (s, d) pairs.

Traffic load:

$$\Lambda_t(X) := \iint_{(s,d)\in B_0(R)\times B_0(R)} \ell\big(X \cap [s,d]\big)\, d\Lambda_\times(s,d)$$

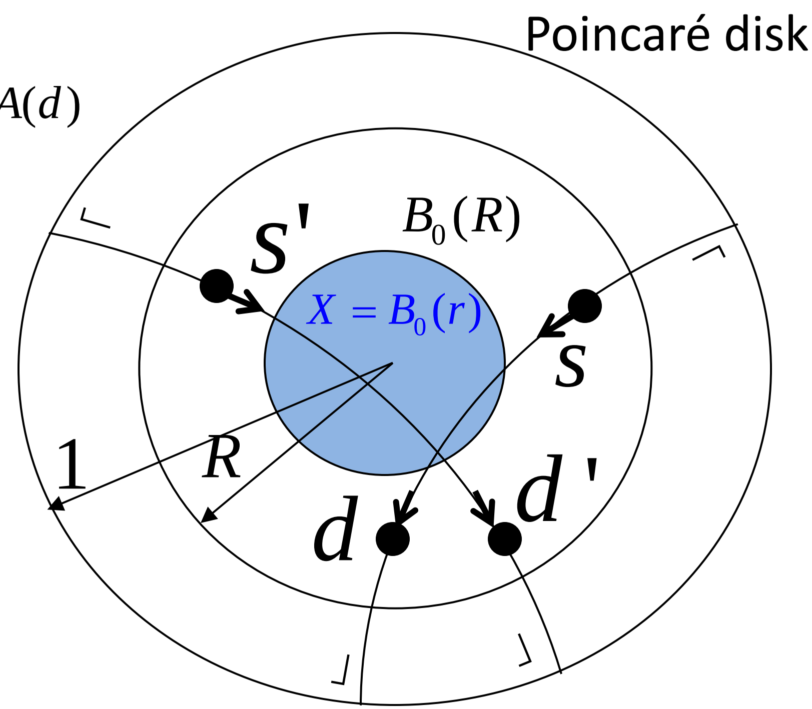
Measure in product space: $d\Lambda_\times(s,d) = dA(s)\times dA(d)$

in Poincaré disk: $dA = \dfrac{4dxdy}{\big(1-(x^2+y^2)\big)^2}$

Density of traffic load:

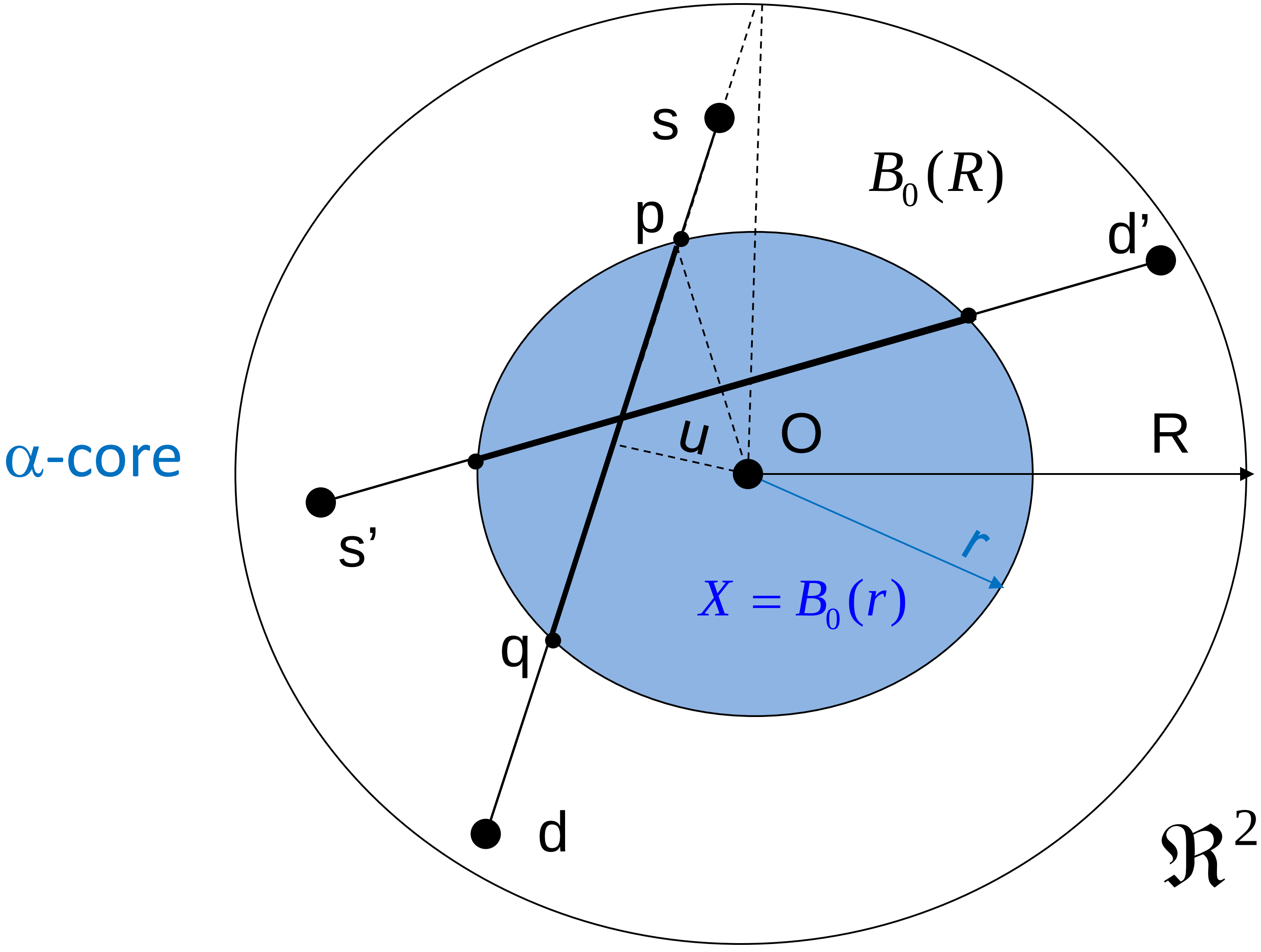
α-core

$$\lambda_t(X) = \frac{\Lambda_t(X)}{\mathrm{vol}\big(B_0(R)\big)} = \alpha$$

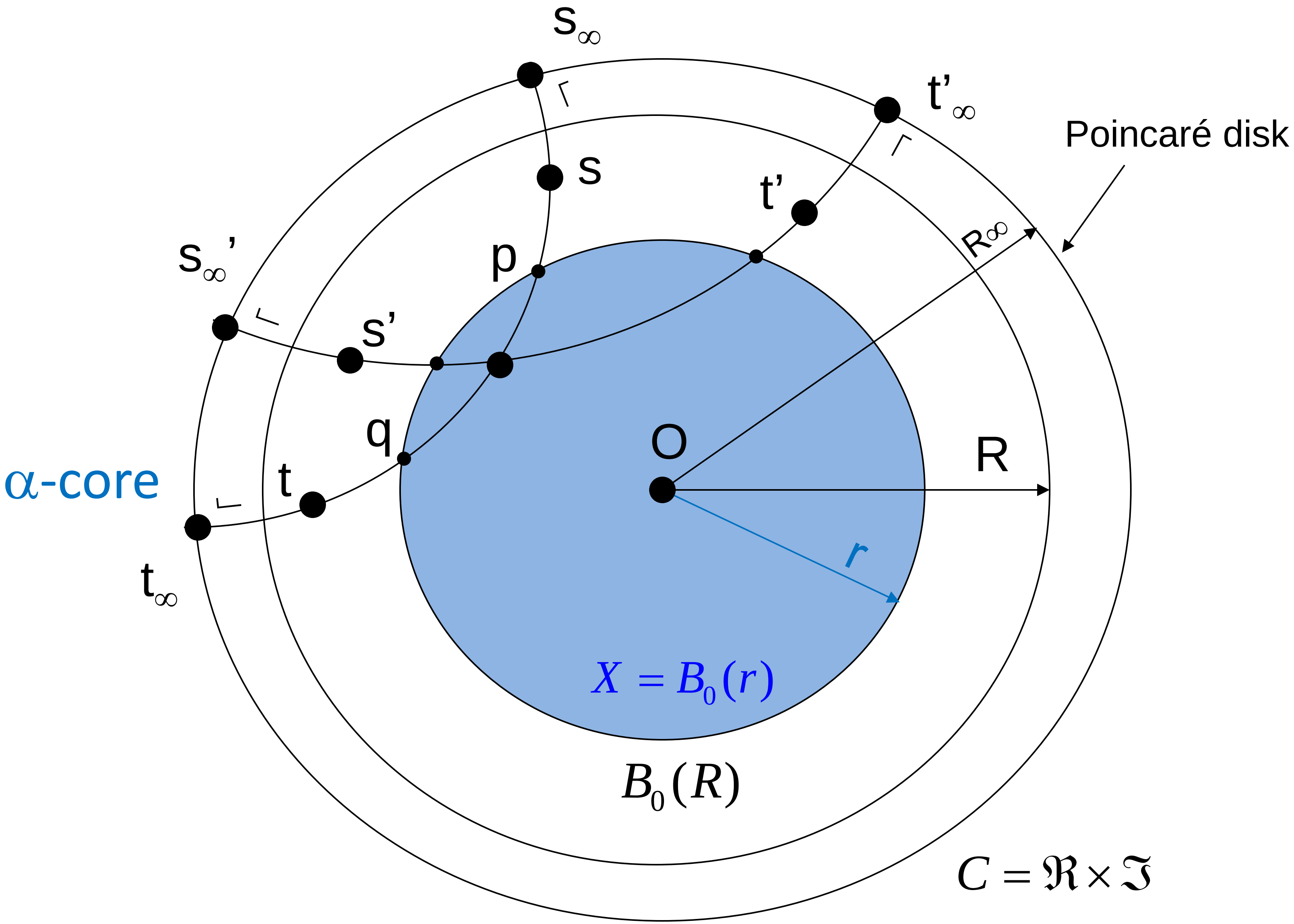


E. Jonckheere, Mingji Lou, F. Bonahon, and Y. Baryshnikov, ``Euclidean versus hyperbolic congestion in idealized versus experimental networks,'' *Internet Mathematics*, vol. 1, number 7, pp. 1-27, 2011.

Details of Euclidean situation



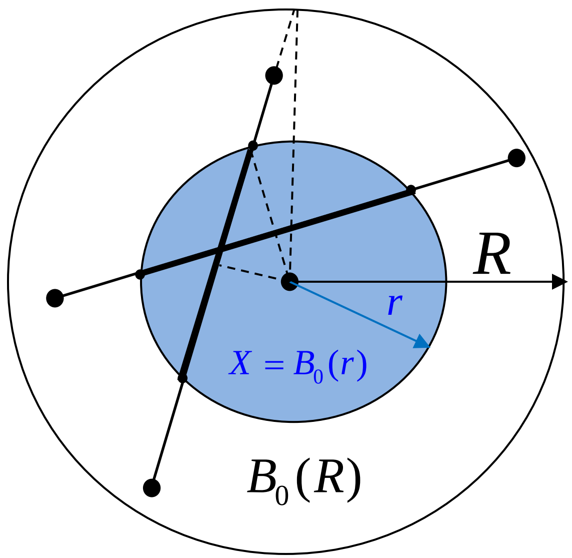
Details of hyperbolic situation



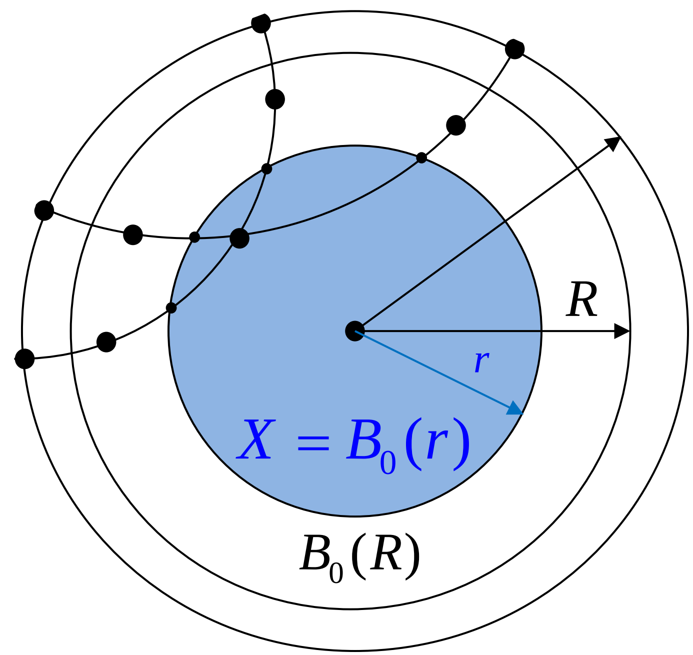
BIG difference between Euclidean and hyperbolic cases

$$\lambda_t(X) \sim c_0(n)\frac{r^n}{R^{n-1}}$$

$$c_{k_1}(n)r^n \le \lambda_t(X) \le c_{k_2}(n)r^n$$



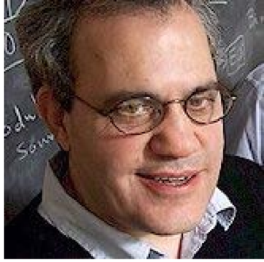
α-core

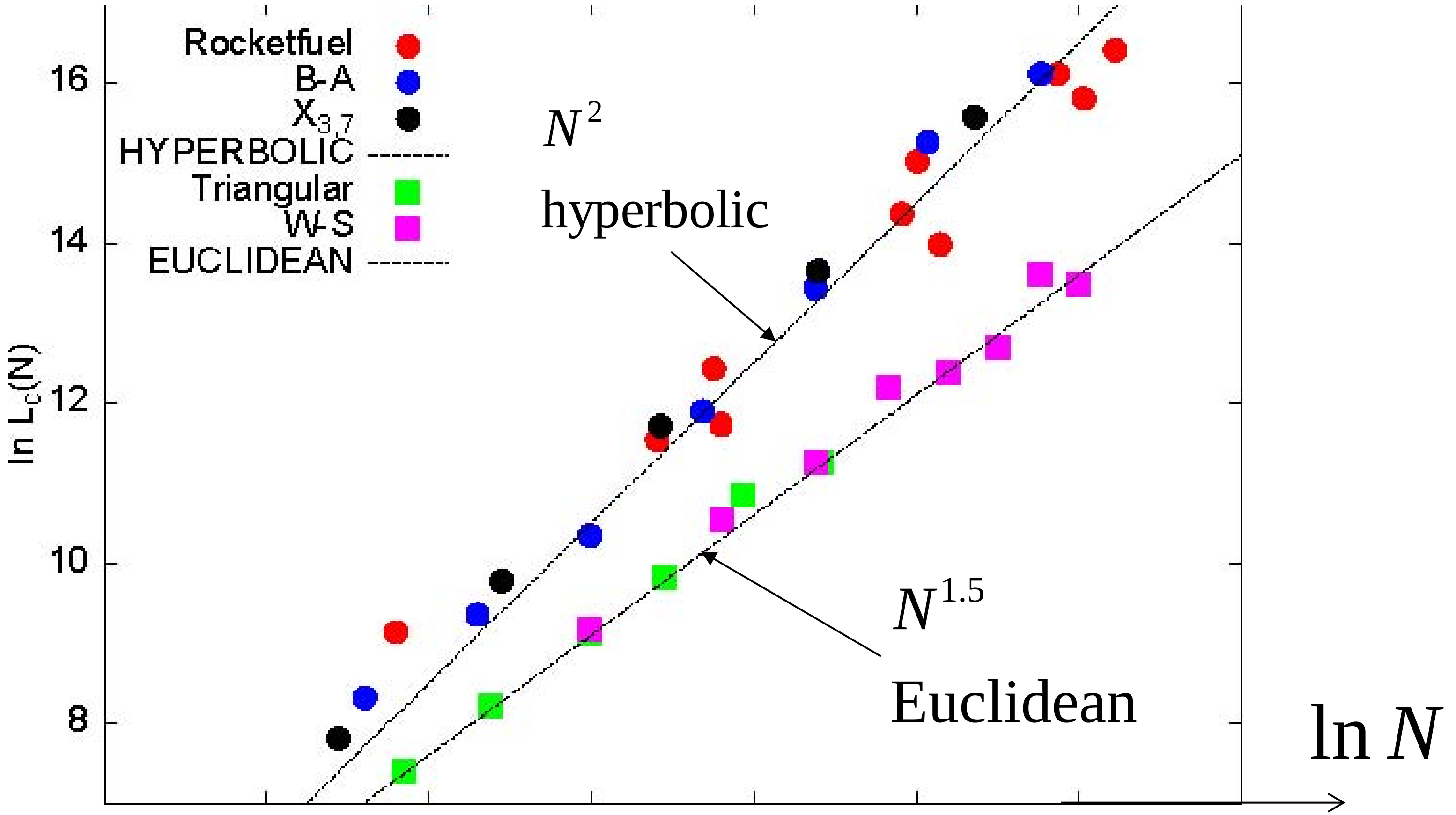


$$\Lambda_t(B_0(r)) = 0\left(\left(\mathrm{vol}\left(B_0(R)\right)\right)^{1.5}\right) \quad (n=2) \qquad \Lambda_t(B_0(r)) = 0\left(\left(\mathrm{vol}\left(B_0(R)\right)\right)^{2}\right)$$

$$\Lambda_t(B_0(r)) = 0\left(\left(\mathrm{vol}\left(B_0(R)\right)\right)^{1+\frac{1}{n}}\right) \quad (\text{general } n) \qquad \Lambda_t(B_0(r)) = 0\left(\left(\mathrm{vol}\left(B_0(R)\right)\right)^{2}\right)$$

Experimental verification of theoretical results by Narayan and Saniee (Bell Labs-Nokia)



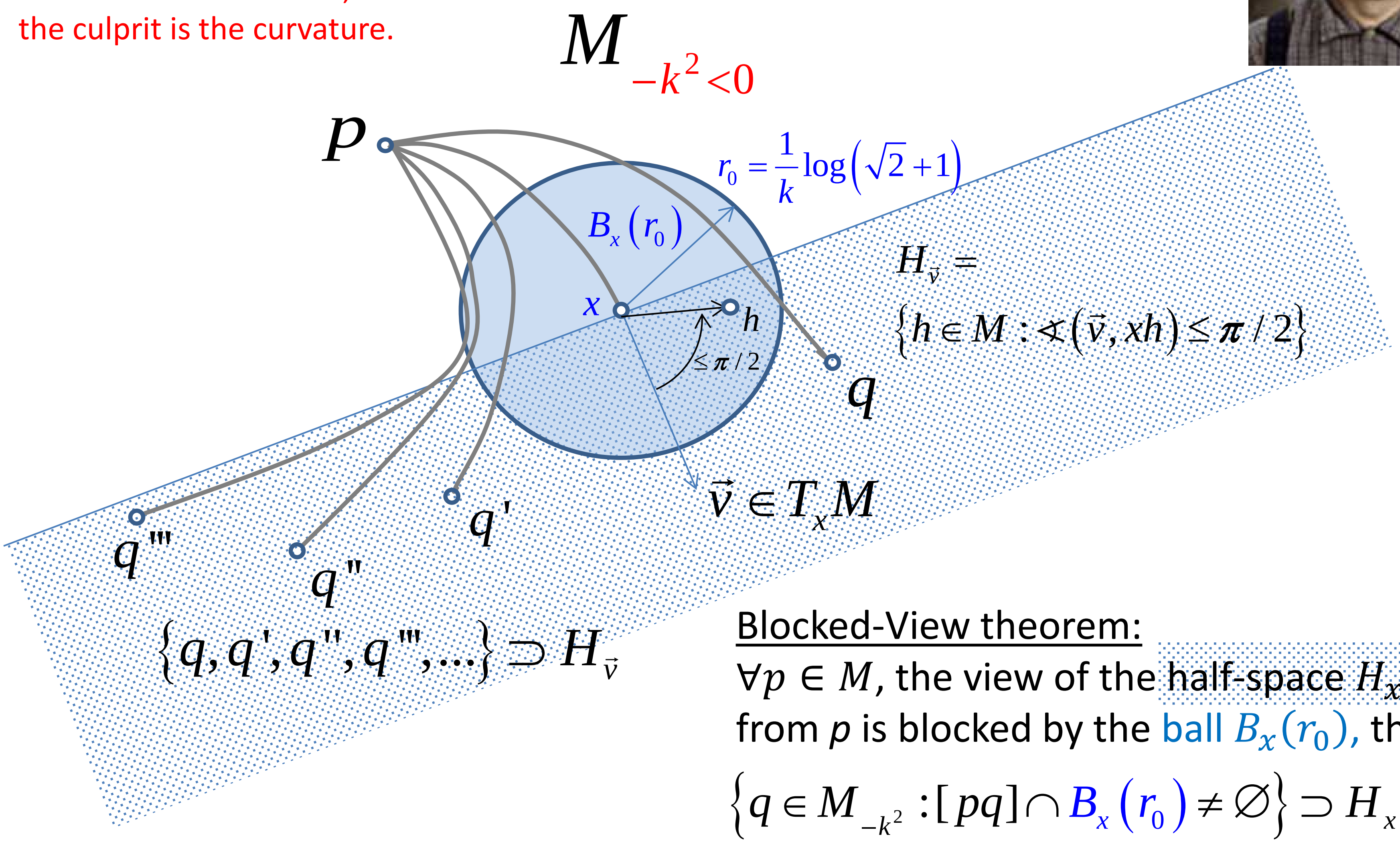


$$N = |V| = \mathrm{vol}\left(B_R(0)\right)$$

O.Narayan and Iraj Saniee, "Large-scale curvature of networks," *Physical Review E (statistical physics), Vol. 84, No. 066108*, Dec. 2011.

Alternative approach to congestion (with F. Bonahon)

For the universal radius,
the culprit is the curvature.



<u>Blocked-View theorem:</u>
$\forall p \in M$, the view of the half-space $H_x(\vec{v})$ from p is blocked by the ball $B_x(r_0)$, that is,

$$\left\{q \in M_{-k^2} : [pq] \cap B_x(r_0) \neq \emptyset\right\} \supset H_x(\vec{v})$$

for the universal radius $r_0 = \frac{1}{k}\log(\sqrt{2}+1)$.

For the congestion density,
the culprit is the dimension!

Define a *fair* density of geodesic estimate:

$$\Phi(M^n) = \sup_{x\in M} \inf_{\vec{v}\in T_x(M)} \frac{\overbrace{\mathrm{vol}(H_x(\vec{v}))}^{\substack{\text{density of geodesics through} \\ \text{ball } B_x(r_0)}}}{\mathrm{vol}(M^n)}$$

Fair-Cut theorem: For a compact, convex manifold M^n of curvature $-k^2 < 0$, the fair congestion estimate is bounded as

$$\frac{1}{n+1} \leq \Phi(M^n) \leq \frac{1}{2}$$

Congestion theorem: For a compact, convex manifold M^n of curvature $-k^2 < 0$, there exist a universal radius $r_0 = \frac{1}{k}\log(\sqrt{5}+1)$ and a point $x \in M^n$, the ***gravity center***, such that the ball $B_x(r_0)$ has congestion density $\Phi(M) \geq \frac{1}{n+1}$.

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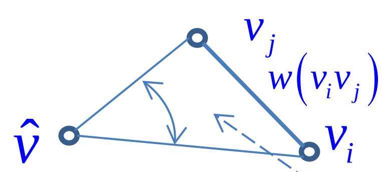
Where the Internet meets Grigori Perelman's proof of the Poincaré conjecture

"Mathematicians make the headlines when they do weird things"
– F. Bonahon

Since points with negative PL curvature, $K(v) = \frac{2\pi - \sum_{i,j} \measuredangle \Delta v_i \hat{v} v_j}{\sum_{ij} A(\Delta v_i v v_j)} < 0$, create congestion,

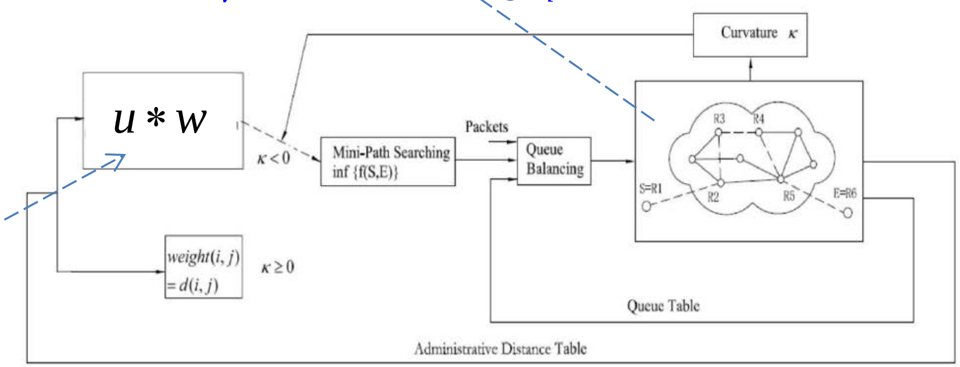
load balancing could be achieved by uniformizing the curvature subject to χ.

This is what Perelman did in his proof of the Poincaré conjecture, *using the Ricci flow.* Here we use the *Yamabe flow.*



Evolution of conformal factors $u: V \times [0,\infty) \to \mathbb{R}$:

$$\frac{du(v_i,t)}{dt} = -K_{u*d}(v_i)\,u(v_i,t), \quad u(v_i,0) = 1$$



Administrative distance table $w(v_i, v_j)$ is modified as

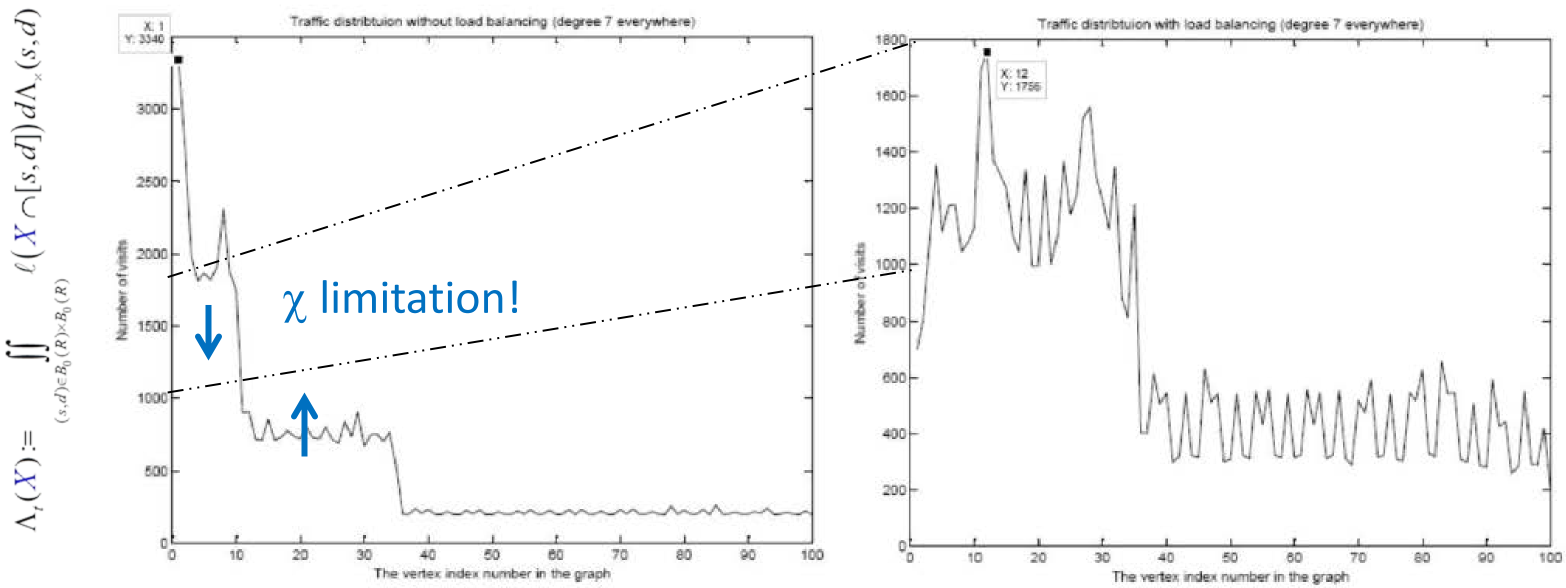
$$u * w(v_i v_j) = u(v_i)\,w(v_i v_j)\,u(v_j)$$

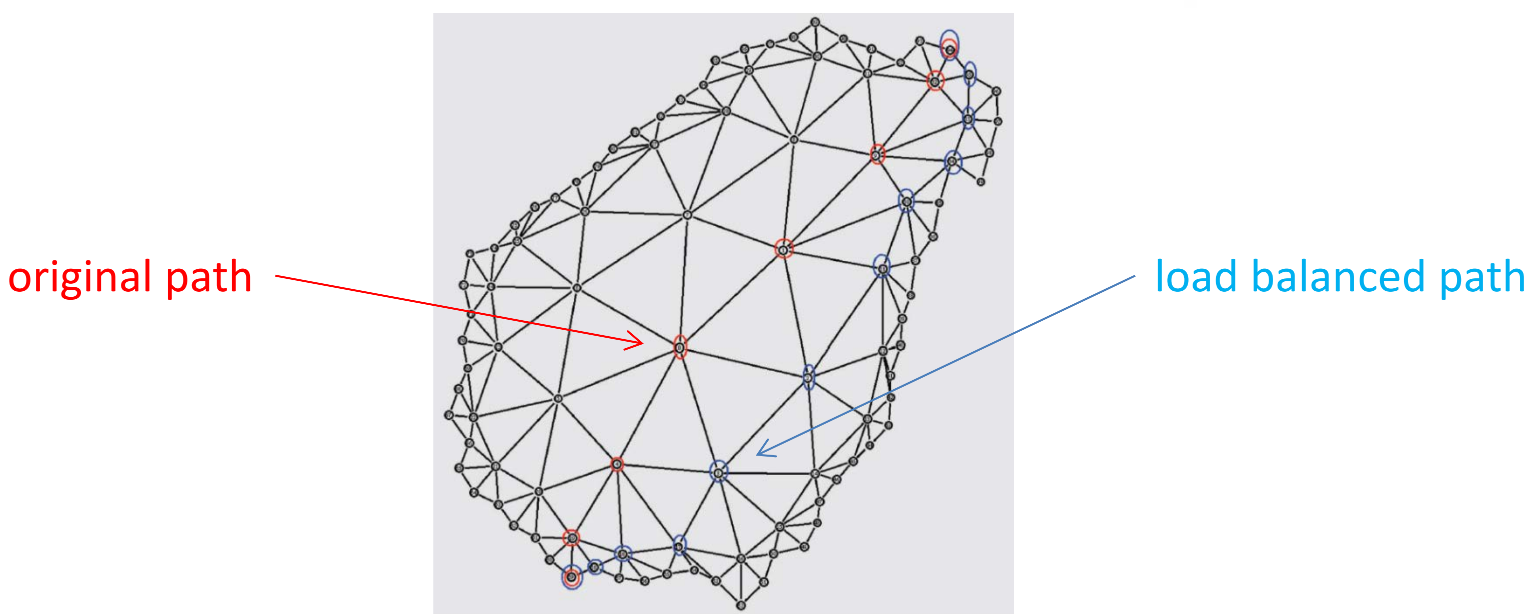
Like Perelman, we encountered singularities when $A(\Delta v_i \hat{v} v_j) = 0$, *removable by edge deletion surgery.*

M. Lou, E. Jonckheere, Y. Baryshnikov, F. Bonahon, and B. Krishnamachari, ``Load Balancing by Network Curvature Control, *International Journal of Computers, Communications and Control (IJCCC),* ISSN 1841-9836, vol.6(1), pp. 134-149, March 2011.

For a PL version of the traffic load $\Lambda_t(X)$, usually referred to as betweenness

$$\Lambda_t(X) := \iint_{(s,d)\in B_0(R)\times B_0(R)} \ell\big(X \cap [s,d]\big)\, d\Lambda_\times(s,d)$$





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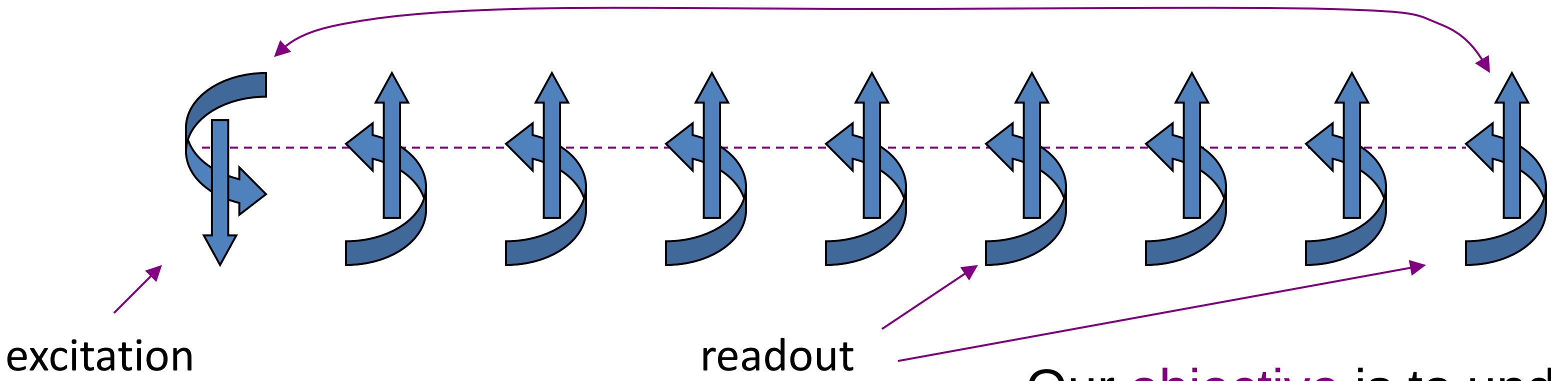
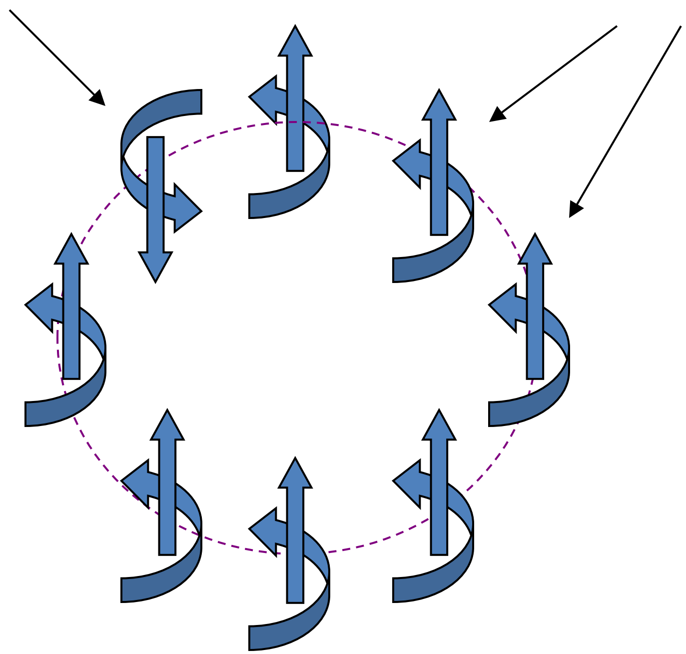
Spin chains and spin rings

Fidelity (or probability) = $\left\langle \downarrow \right| e^{-iHt_f} \left| \downarrow \right\rangle$





- Our objective is to understand how excitation is transmitted in spin chains and spin rings of the Heisenberg or XX type in the first excitation subspace.
- Our approach is geometrical:
    - Define a distance
    - Understand the geometry of the spin network for the given distance
- What does the geometry tell us?
- What are the applications?

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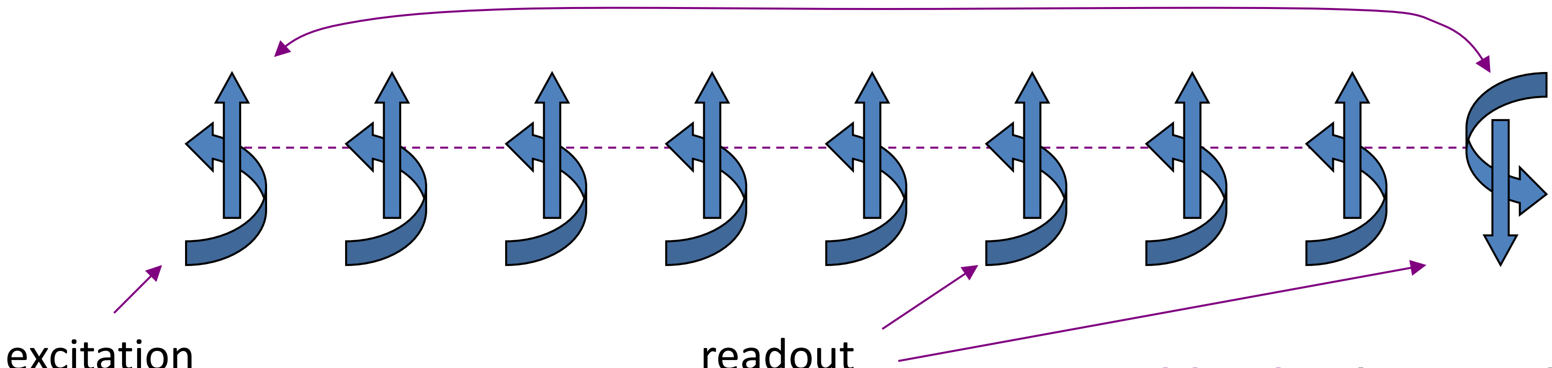

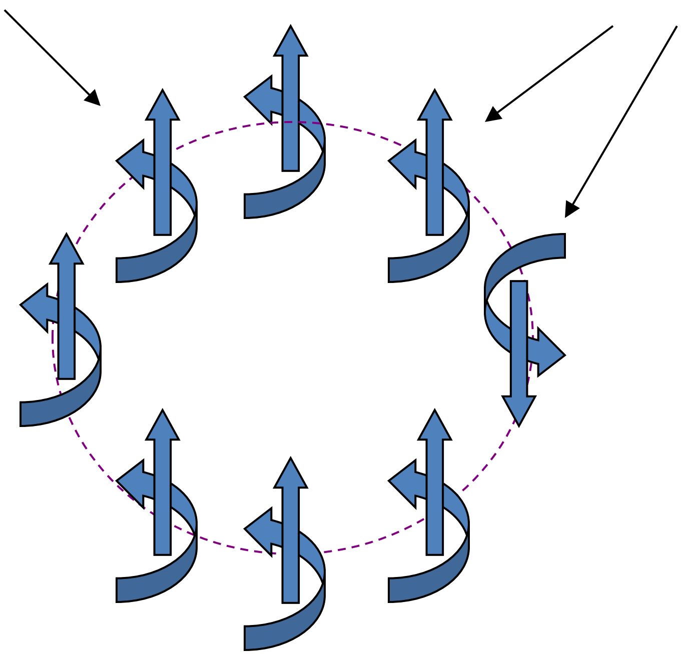

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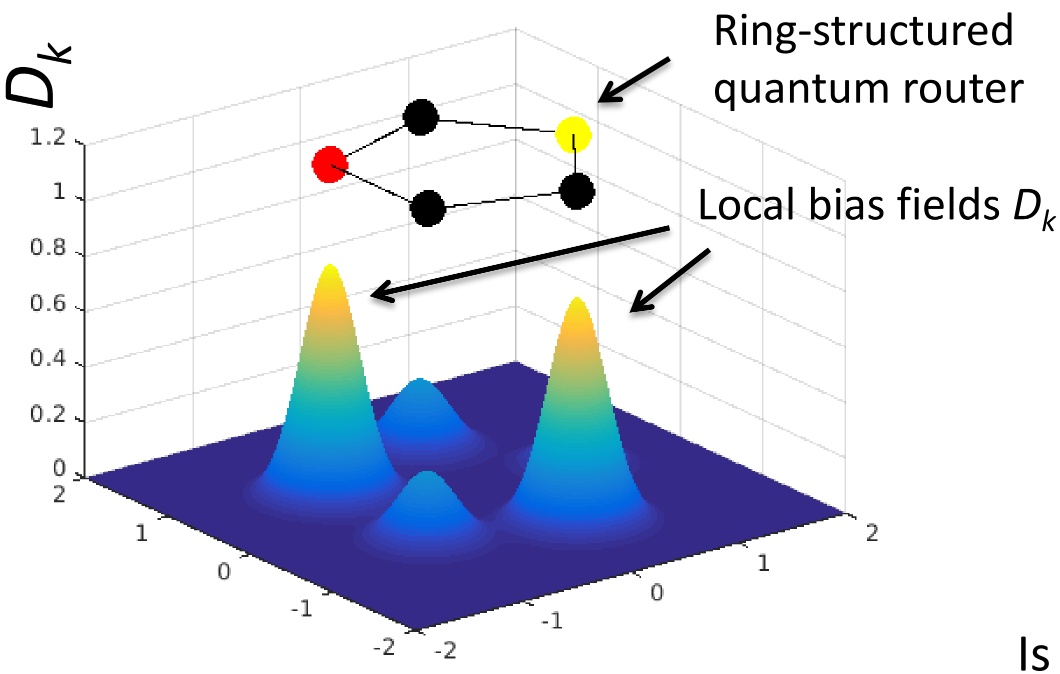
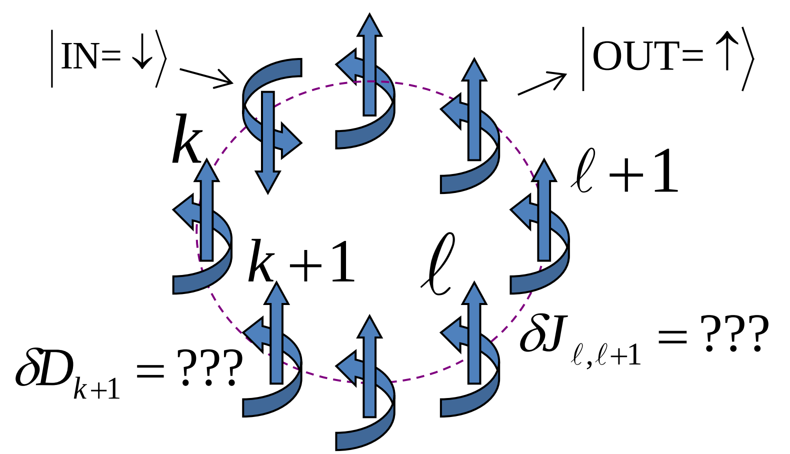


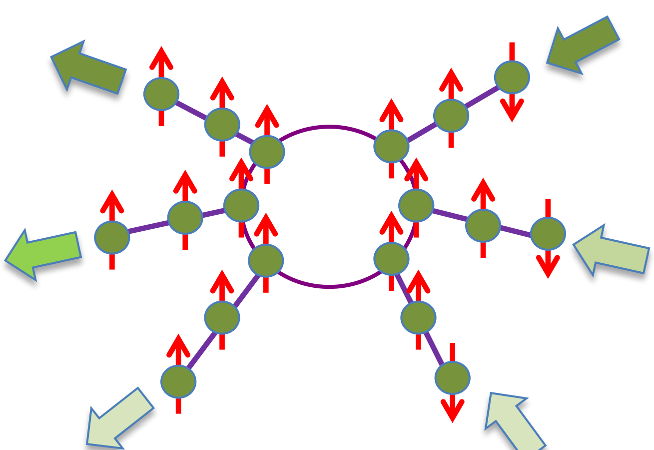
Robustness issue:

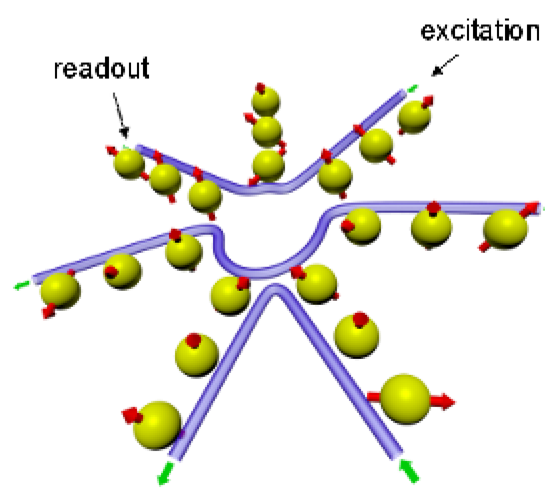


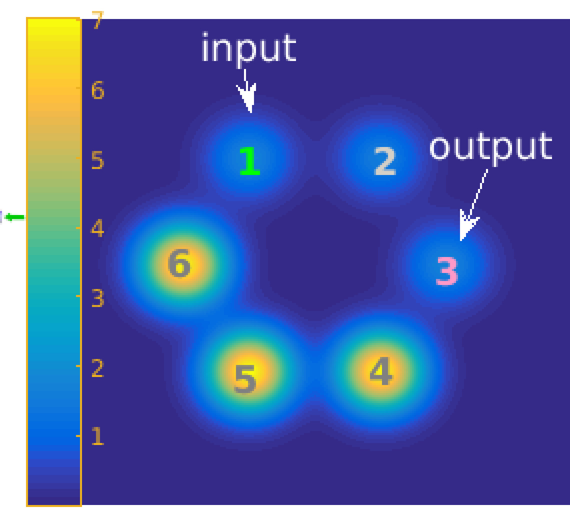
Is the design robust against δD and δJ?

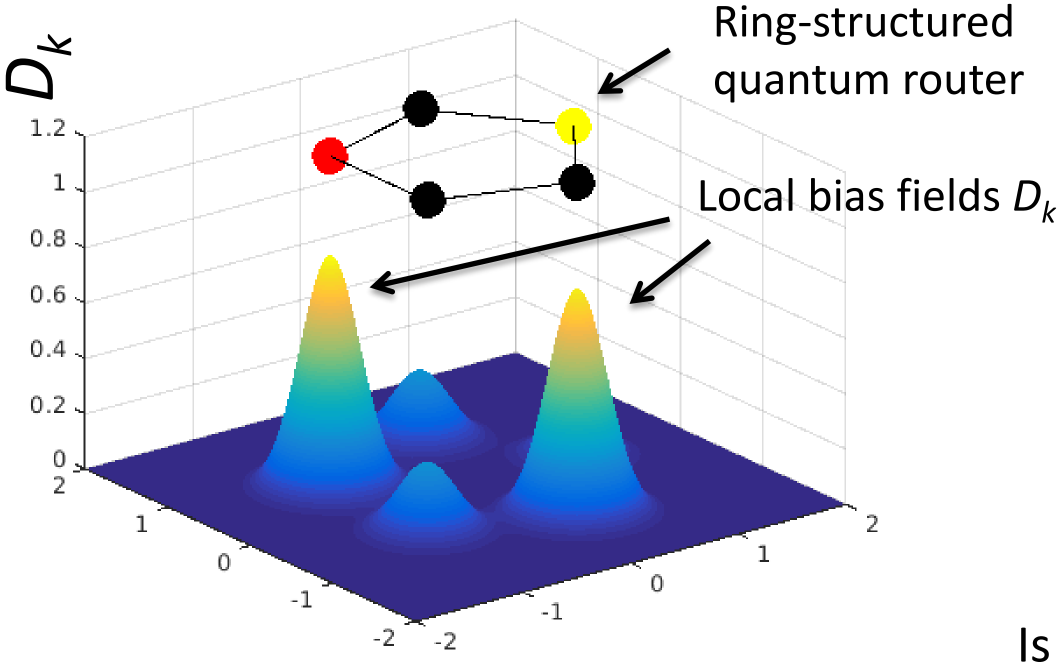
Concept of a “quantum router”

Given ($|\mathrm{IN}\rangle$, $|\mathrm{OUT}\rangle$) pair, find biases so as to favor the specified transmission









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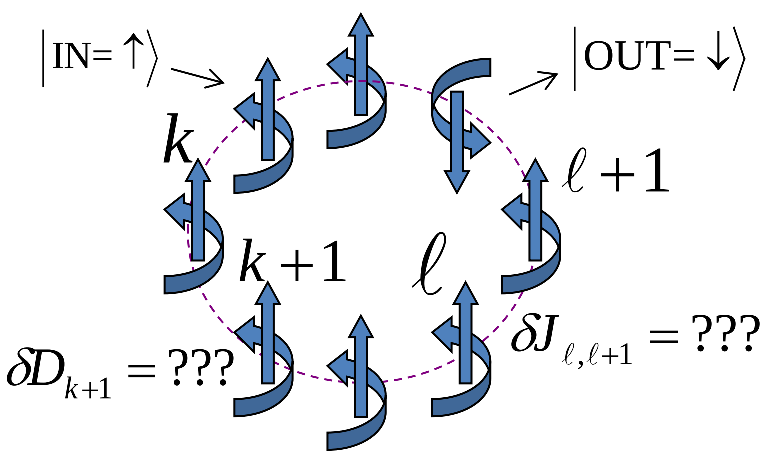

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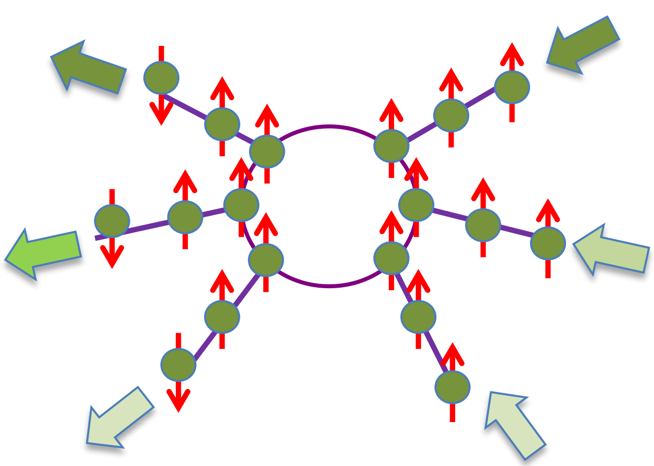
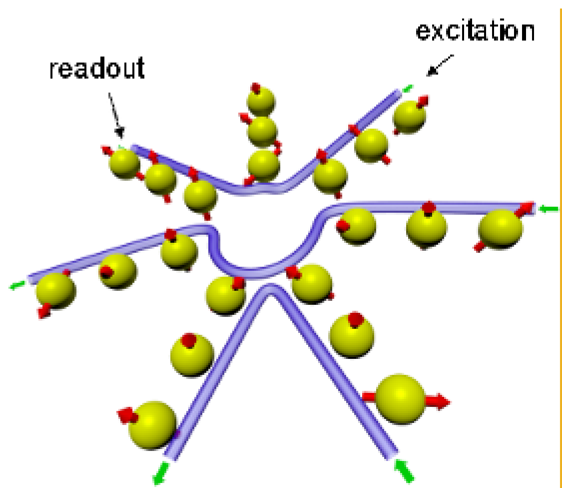
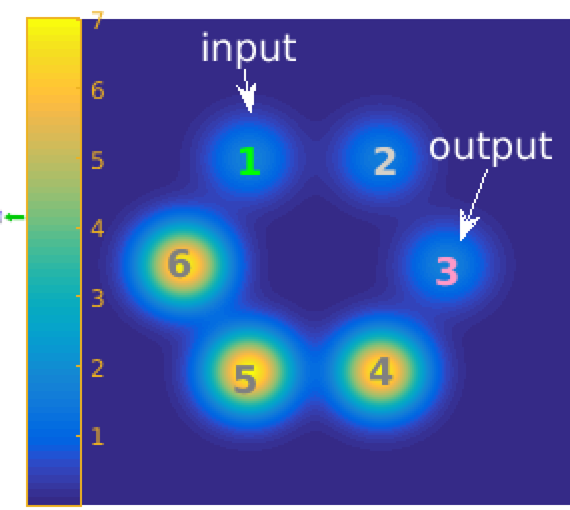
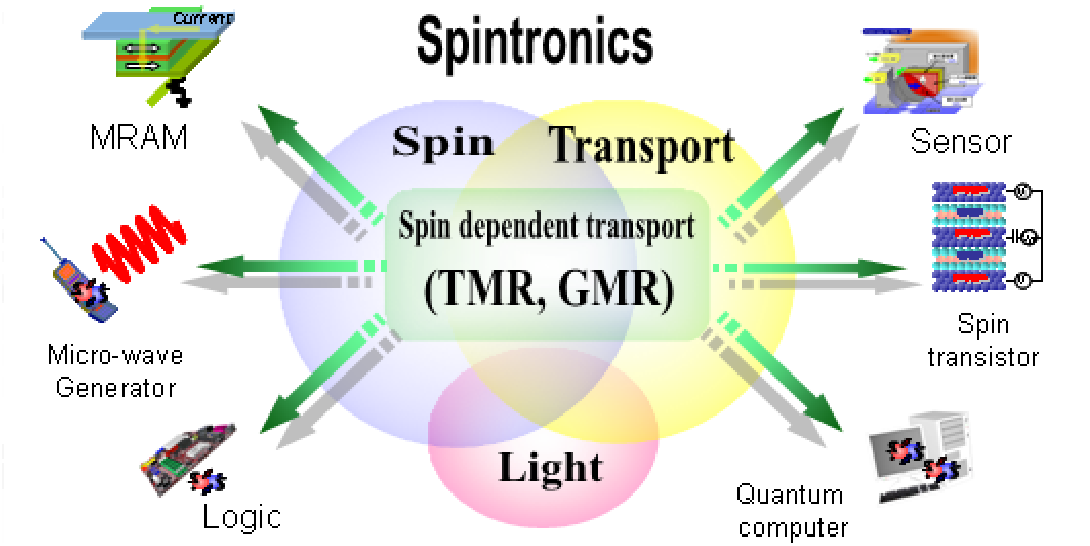




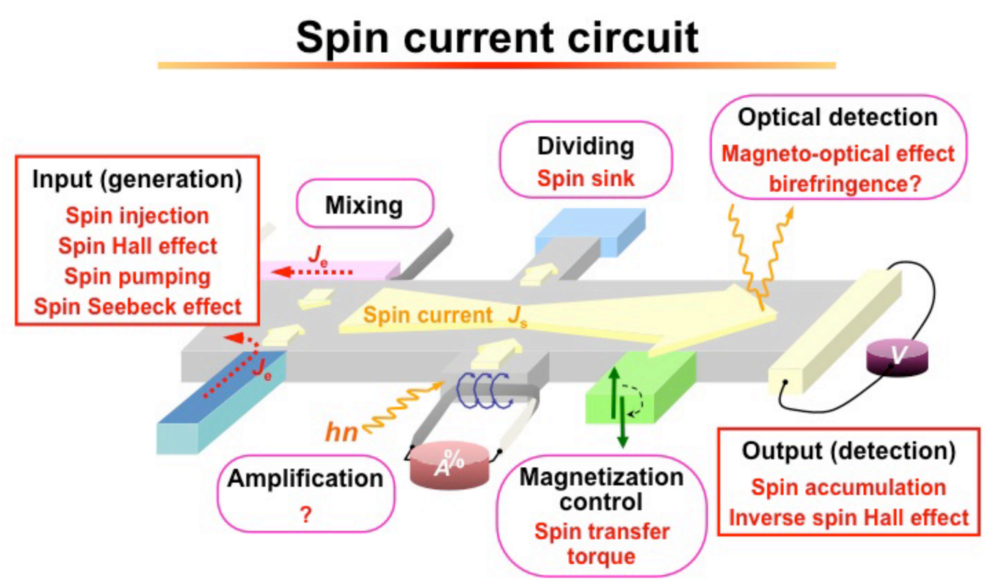


Spintronics devices

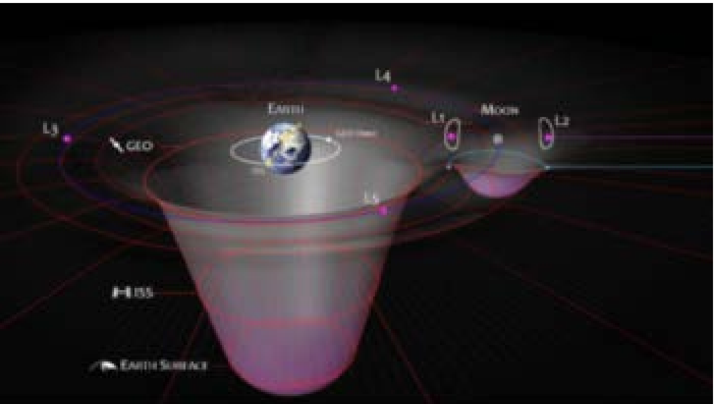


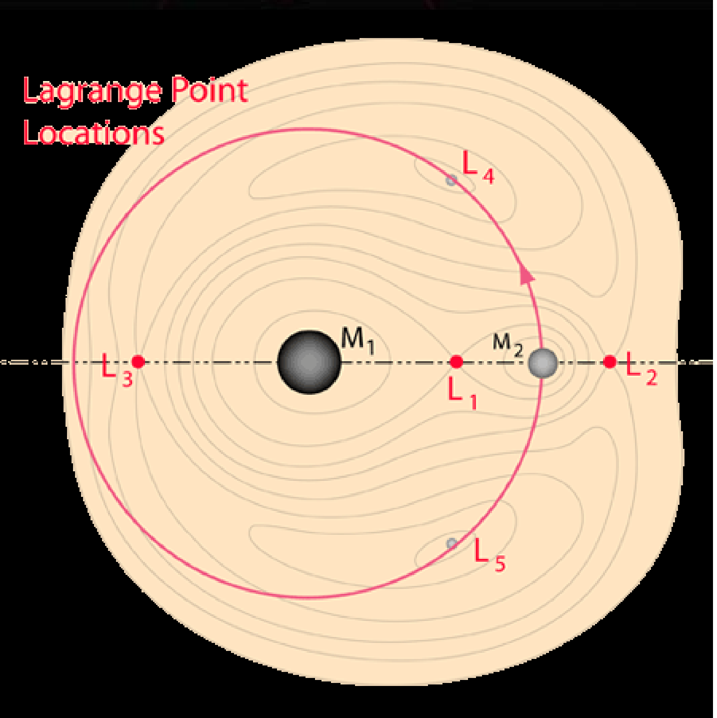
Spintronics, or spin electronics, refers to the study of the role played by electron (and more generally nuclear) spin in solid state physics.

Physicists are trying to exploit the spin of the electron rather than its charge to create a new generation of spintronics devices, smaller, more versatile than silicon chips.



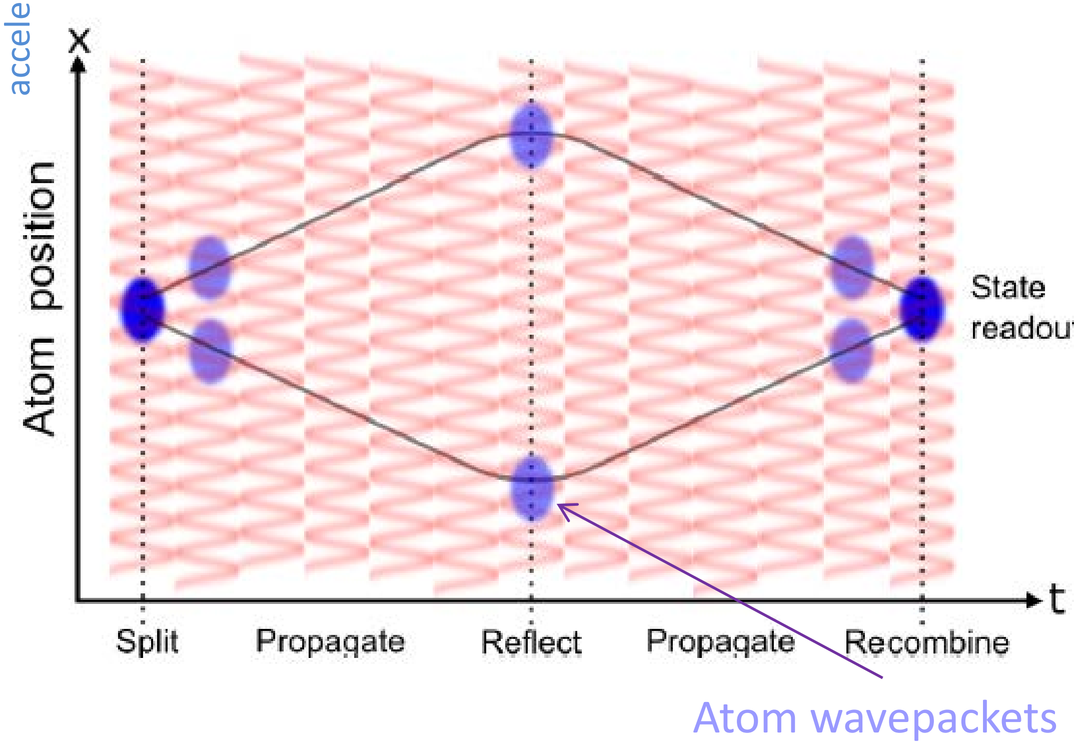
Ultra-cold atom optical lattice: Navigating in the Cislunar space by Shaken Lattice Interferometry inertial sensing





acceleration

$$H(x,t) = -\frac{\hat{p}^2}{2m} - \frac{V_0}{2}\cos\left(2k_L x + \phi(t)\right) + m a_x x$$



C. Weidner, “Shaken Lattice Interferometry,” Ph.D. dissertation, Univ. Colorado, Boulder, 2018.
F. Ariaei, E. Jonckheere and S. Bohacek, “Tracking Trojan asteroid in periodic and quasi-periodic orbits around the Jupiter Lagrange points using LDV techniques,” *Physics and Control,* St. Petersburg, Russia, 2003, pp. 100-105.

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Maximum state transition probability

$$p\left(|j,t\rangle,|i,0\rangle\right)=\left|\langle j|e^{-\imath Ht}|i\rangle\right|^2 \quad (\hbar=1)$$

solution $\psi(t)$ to Schrödinger's equation with initial quantum state *i*

terminal or desired condition

initial condition

$$=\left|\langle j|e^{-\imath\sum_{k=1}^{N}\lambda_k|v_k\rangle\langle v_k|t}|i\rangle\right|^2$$

Eigen-expansion of Hamiltonian $H_{N\times N}$ *in the single excitation subspace!*

$$=\left|\sum_{k=1}^{N}\langle j|v_k\rangle\langle v_k|i\rangle e^{-\imath\lambda_k t}\right|^2$$

$$\leq\left|\sum_{k=1}^{N}\left|\langle j|v_k\rangle\langle v_k|i\rangle\right|\right|^2 =: p_{\max}(i,j)$$

$1,000,000 question: Can we have equality? $\sup_{t\geq 0} p\left(|j,t\rangle,|i,0\rangle\right)=p_{\max}(i,j)$

Can a quantum network be made a metric space?

Define

$$d(i,j) = \log \frac{1}{p_{\max}(i,j)}$$

1) Do we have the triangle inequality $d(i,j) \leq d(i,k) + d(k,j)$?
   a) On a *uniform* spin **ring**, the triangle inequality has been ***proved!***
   b) On a *uniform* spin **chain**, the triangle inequality has been computationally verified up to order 500. On a *nonumiform* spin **chain**, we observed violations.
2) Do we have $d(i,j) > 0$ for $i \neq j$?
   a) Yes, for a ring of odd size N (metric space)
   b) No, for a ring of even size N (pseudo-metric space)
      i. Yes, after anti-podal spin identification (metric space)
   c) No, in general, for a chain: "good news/bad news!"

P. Bogdan, E. Jonckheere, and S. Schirmer, ``Multi-fractal Geometry of Finite Networks of Spins: Nonequilibrium Dynamics beyond Thermalization and Many-Body-Localization," *Chaos, Solitons & Fractals,* vol. 103, pp. 622-631, October 2017. arXiv:1608.08192 [quant-ph].
E. Jonckheere, F. C. Langbein, and S. G. Schirmer, "Curvature of quantum rings," *International Symposium on Control, Communications and Signal Processing,* Rome, Italy, May 02-04, 2012.

Reachability of maximum transition probability

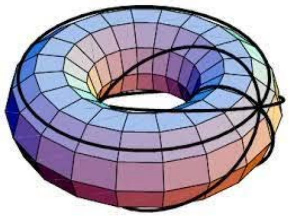
$$p\left(\left| j,t\right\rangle ,\left| i,0\right\rangle\right) = \left| \sum_{k=1}^{N} \left\langle j \middle| v_k \right\rangle \left\langle v_k \middle| i \right\rangle e^{-\imath \lambda_k t} \right|^2 \leq \left| \sum_{k=1}^{N} \left| \left\langle j \middle| v_k \right\rangle \left\langle v_k \middle| i \right\rangle \right| \right|^2 =: p_{\max}(i, j)$$

Clearly, $p_{\max}(i,j)$ can be reached if there exists a time t, large enough, such that

$$-\lambda_k t = (2m+1)\pi, \text{ if } \operatorname{sign}\left(\left\langle j \middle| v_k \right\rangle \left\langle v_k \middle| i \right\rangle\right) = -1$$

$$-\lambda_k t = (2m)\pi, \quad \text{ if } \operatorname{sign}\left(\left\langle j \middle| v_k \right\rangle \left\langle v_k \middle| i \right\rangle\right) = +1$$

We already perceive



- The flow on the torus: $\frac{d\tilde{x}}{dt} = -\text{diag}\{\lambda_1, \lambda_2, \dots \lambda_N\} \bmod 2\pi$, $\tilde{x}(0) = 0$
- The simultaneous Diophantine approximation: $\frac{\lambda_k}{\pi} \approx \frac{p_k}{q}$
- The Lenstra-Lenstra-Lovasz (LLL) algorithm: $\left| p_k - \frac{\lambda_k}{\pi} q \right| < \epsilon$

E. Jonckheere, S. Schirmer, and F. Langbein, ``Information transfer fidelity in spin networks and ring-based quantum routers,'' *Quantum Information Processing (QINP),* DOI 10.1007/s11128-015-1136-4, 2015.
Available at http://ee.usc.edu/~jonckhee and arXiv:submit/1359959 [quant-ph] 24 Sep 2015.

We forgot the global phases!!!

much more likely to happen!

$$p\left(|j,t\rangle,|i,0\rangle\right)=\left|\sum_{k=1}^{N}\left\langle e^{\imath\phi_j} j \middle| v_k\right\rangle\left\langle v_k \middle| e^{\imath\phi_i} i\right\rangle e^{-\imath\lambda_k t}\right|^2 = \left|\sum_{k=1}^{N}\left|\langle j|v_k\rangle\langle v_k|i\rangle\right|\right|^2 =: p_{\max}(i,j)$$

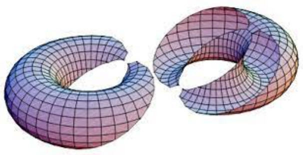
Clearly, $p_{\max}(i,j)$ can be reached if there exists a time *t*, large enough, such that

$$-\lambda_k t + (\phi_i - \phi_j) = (2m+1)\pi, \text{ if } \operatorname{sign}\left(\langle j|v_k\rangle\langle v_k|i\rangle\right) = -1$$

$$-\lambda_k t + (\phi_i - \phi_j) = (2m)\pi, \quad \text{ if } \operatorname{sign}\left(\langle j|v_k\rangle\langle v_k|i\rangle\right) = +1$$

But it is unclear what becomes of

- The flow on the torus: $\frac{d\tilde{x}}{dt} = -\text{diag}\{\lambda_1, \lambda_2, \dots \lambda_N\} \bmod 2\pi\,,\ \tilde{x}(0) = 0$
- The simultaneous Diophantine approximation: $\frac{\lambda_k}{\pi} \approx \frac{p_k}{q}$
- The Lenstra-Lenstra-Lovasz (LLL) algorithm: $\left|p_k - \frac{\lambda_k}{\pi} q\right| < \epsilon$



E. Jonckheere, S. Schirmer, and F. Langbein, ``Information transfer fidelity in spin networks and ring-based quantum routers," *Quantum Information Processing (QINP),* DOI 10.1007/s11128-015-1136-4, 2015.
Available at http://ee.usc.edu/~jonckhee and arXiv:submit/1359959 [quant-ph] 24 Sep 2015.

Basic attainability condition:

$$e^{-\imath\lambda_k t} = \underbrace{\mathrm{sign}\left(\langle j \mid v_k\rangle\langle v_k \mid i\rangle\right)}_{s_k} e^{\imath\phi}, \quad \phi = \phi_i - \phi_j$$

Recipe

$$s_k = \exp\left[-\imath\pi\left(2n_k + \frac{1}{2}(s_k - 1)\right)\right], \quad n_k \in \mathbb{Z}, \quad \forall k$$

Substitute and take log

$$-\imath\lambda_k t = -\imath\pi\left(2n_k + \frac{1}{2}(s_k - 1)\right) + \imath\phi$$

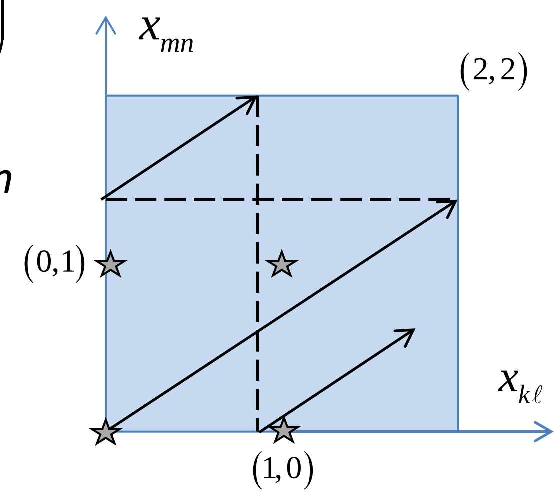
Get rid of ϕ by appealing to other modes:

$$(\lambda_k - \lambda_\ell)t = \pi\left(2(n_k - n_\ell) + \frac{1}{2}(s_k - s_\ell)\right)$$

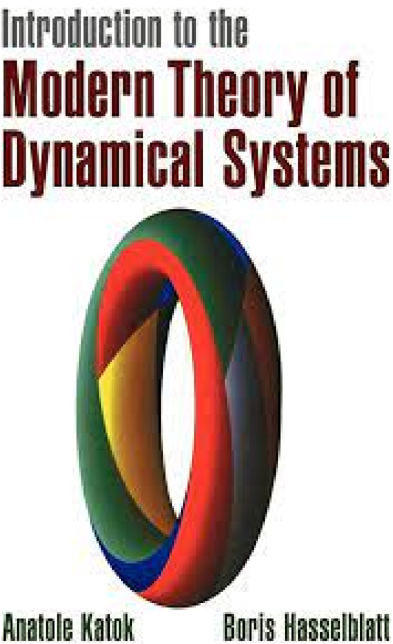
Back to flow on torus,

where the global phase has tacitly been taken into consideration

$$\underbrace{\frac{\lambda_k - \lambda_\ell}{\pi}}_{\omega_{k\ell}} t = \frac{1}{2}(s_k - s_\ell) \bmod 2$$



Definition: A flow or translation on the torus $\mathbb{T}^2$ is said to be ***minimal*** iff the orbit of every initial point is everywhere dense in $\mathbb{T}^2$.



Theorem: The flow on the torus is *minimal* iff the $\{\omega_{k,\ell}\}_{k,\ell=1:N}$ are linearly independent over the rationals $\mathbb{Q}$.

$$\boldsymbol{\omega}_{k\ell} t = \frac{1}{2}(s_k - s_\ell) \bmod 2, \quad \text{flow on torus}, \quad t \in \mathbb{R}$$

$$\boldsymbol{\theta}_{k\ell} \boldsymbol{\tau} = \frac{1}{2}(s_k - s_\ell) \bmod 2, \quad \text{translation on torus}, \quad \boldsymbol{\tau} \in \mathbb{Z}$$

$$t = \frac{2\boldsymbol{\tau}}{\boldsymbol{\omega}_{m,n}}, \quad \boldsymbol{\tau} \in \mathbb{Z}, \quad m, n = \text{dark states}, \quad \boldsymbol{\theta}_{k,\ell} = \frac{\boldsymbol{\omega}_{k,\ell}}{\boldsymbol{\omega}_{m,n}}$$

Theorem: The translation on the torus is *minimal* iff the set $\left\{1, \{\theta_{k,\ell}\}_{k,\ell=1:N}\right\}$ is linearly independent over $\mathbb{Q}$.

Dark states

$$H = H_0 + H_D = \sum_{k=1}^{N} \lambda_k \Pi_k, \quad \Pi_k = |v_k\rangle\langle v_k| \text{ (if eigenvalue is simple)}$$

Perfect state transfer (or super-optimality) by control H_D:

$$\left| \sum_{k=1}^{N} \langle j | e^{-\imath (H_0 + H_D) t_f} | i \rangle \right| = 1 \Rightarrow \boxed{\langle j | v_k \rangle = \pm \langle v_k | i \rangle}$$

11-ring: $|i = 1\rangle \mapsto |j = 3\rangle$, that is, $\psi(0) = e_1 \mapsto \psi(t) = e_3$

V=

		+1	-1	+1	-1	+1				
0.0000	**-0.0000**	0.6914	-0.7014	0.0064	0.0893	0.1481	**- 0.0000**	**0.0000**	**-0.0000**	**-0.0000**
-0.0000	0.0000	-0.1714	0.0000	0.6098	0.0000	0.7738	0.0000	0.0000	-0.0000	-0.0000
-0.0000	**-0.0000**	0.6914	0.7014	0.0064	-0.0893	0.1481	**0.0000**	**0.0000**	**0.0000**	**-0.0000**
0.0000	0.0000	-0.0854	-0.0893	-0.5604	-0.7014	0.4227	0.0000	0.0000	0.0009	-0.0010
-0.0009	-0.0009	0.0001	0.0001	0.0008	0.0009	-0.0006	0.0000	0.0000	0.6453	-0.7639
0.7070	0.7070	0.0000	0.0000	0.0000	0.0000	-0.0000	-0.0103	-0.0100	0.0008	-0.0010
-0.0103	-0.0100	-0.0000	-0.0000	-0.0000	-0.0000	0.0000	-0.7070	-0.7070	0.0000	-0.0000
0.0103	-0.0100	-0.0000	0.0000	-0.0000	0.0000	0.0000	0.7070	-0.7070	-0.0000	-0.0000
-0.7070	0.7070	0.0000	-0.0000	0.0000	-0.0000	-0.0000	0.0103	-0.0100	-0.0010	-0.0008
0.0009	-0.0009	0.0001	-0.0001	0.0008	-0.0009	-0.0006	-0.0000	0.0000	-0.7639	-0.6453
-0.0000	0.0000	-0.0854	0.0893	-0.5604	0.7014	0.4227	-0.0000	0.0000	-0.0010	-0.0009

dark states symmetry! **dark states**

S. Schirmer, E. Jonckheere, and F. Langbein, ``Design of feedback control laws for information transfer in spintronics networks," *IEEE Transactions on Automatic Control,* vol. 63, No. 8, pp. 2523-2536, August 2018. Available at http://ee.usc.edu/~jonckhee and arXiv:1607.05294.

Taking global phases into consideration

Equality can be achieved!

$$p\left(\left| j,t\right\rangle ,\left| i,0\right\rangle \right)=\left|\sum_{k=1}^{N}\left\langle e^{\imath\phi_j} j \middle| v_k\right\rangle\left\langle v_k \middle| e^{\imath\phi_i} i\right\rangle e^{-\imath\lambda_k t}\right|^2 = \left|\sum_{k=1}^{N}\left|\left\langle j \middle| v_k\right\rangle\left\langle v_k \middle| i\right\rangle\right|\right|^2 =: p_{\max}(i,j)$$

Clearly, $p_{\max}(i,j)$ can be reached if there exists a time $\tau \in \mathbb{N}$, large enough, such that

$$\underbrace{\frac{\lambda_k - \lambda_\ell}{\pi}}_{\theta_{k\ell}}\tau = \frac{1}{2}\left(s_k - s_\ell\right)\bmod 2$$

Now, we have the three-fold interpretation:

- The translation on the torus: $\theta_{k\ell}(\tau+1)=\theta_{k\ell}(\tau)+\big((s_k-s_\ell)\mathrm{mod}2\big)/2$
- The simultaneous Diophantine approximation: $\theta_{k\ell}\approx\frac{p_{k\ell}}{q}$
- The Lenstra-Lenstra-Lovasz (LLL) algorithm: $|p_{k\ell}-\theta_{k\ell}q|<\epsilon$

E. Jonckheere, S. Schirmer, and F. Langbein, ``Information transfer fidelity in spin networks and ring-based quantum routers," *Quantum Information Processing (QINP),* DOI 10.1007/s11128-015-1136-4, 2015.
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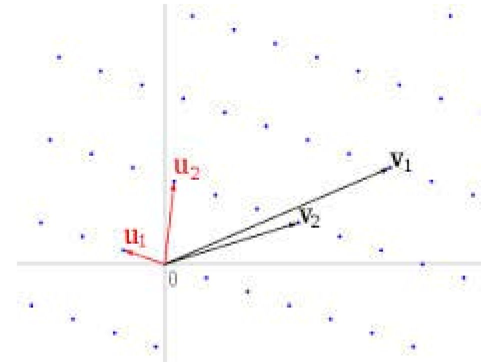
Lenstra-Lenstra-Lovasz (LLL or L^3) algorithm



The LLL-algorithm find (the) short(est) basis vectors in a lattice.



$$\theta = \mathrm{col}\left\{\theta_{k\ell} : k \neq \ell\right\}$$

$$\underbrace{\overbrace{\begin{pmatrix} I_{\bar{N}\times\bar{N}} & -\theta \\ 0_{1\times\bar{N}} & s \end{pmatrix}}^{\text{lattice base matrix } B(s)} \begin{pmatrix} p \\ q \end{pmatrix}}_{\text{a vector in the lattice } B(s)Z^{\bar{N}+1}} = \begin{pmatrix} p-\theta q \\ sq \end{pmatrix}$$

Diophantine approximation error that must be kept small.

Hence let s↓0 and a (the) short(est) vector gives a good (best) simultaneous Diophantine approximation.

$$B^*(s) = \overbrace{\begin{pmatrix} b^*_{1:\bar{N},1} & b^*_{1:\bar{N},2} & \cdots & b^*_{1:\bar{N},N+1} \\ b^*_{\bar{N}+1,1} & b^*_{\bar{N}+1,2} & \cdots & b^*_{\bar{N}+1,\bar{N}+1} \end{pmatrix}}^{\text{Reduced basis of short(est) vectors}}$$

$$q = \frac{1}{s}\left(B^*(s)\right)_{\bar{N}+1,1}$$

$$p_i = \left(B^*(s)\right)_{1:\bar{N},1} + \theta_i q$$

X-weighted LLL-algorithm

X-weighted Diophantine approximation error

$$\overbrace{\begin{pmatrix} X & -X\theta \\ 0_{1\times\overline{N}} & s \end{pmatrix}}^{B(s,X)}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} X(p-\theta q) \\ sq \end{pmatrix}, \quad X = \begin{pmatrix} x_{1,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_{\overline{N},\overline{N}} \end{pmatrix}$$

a vector in the lattice $B(s,X)Z^{\overline{N}+1}$

Reduced basis of short(est) vectors

$$B^*(s,X) = \begin{pmatrix} b^*_{1:\overline{N},1}(s,X) & b^*_{1:\overline{N},2}(s,X) & \cdots & b^*_{1:\overline{N},N+1}(s,X) \\ b^*_{\overline{N}+1,1}(s,X) & b^*_{\overline{N}+1,2}(s,X) & \cdots & b^*_{\overline{N}+1,\overline{N}+1}(s,X) \end{pmatrix}$$

$$q = \frac{1}{s}\left(B^*(s,X)\right)_{\overline{N}+1,1}$$

$$p_i = \frac{1}{x_{ii}}\left(B^*(s,X)\right)_{1:\overline{N},1} + \theta_i q$$

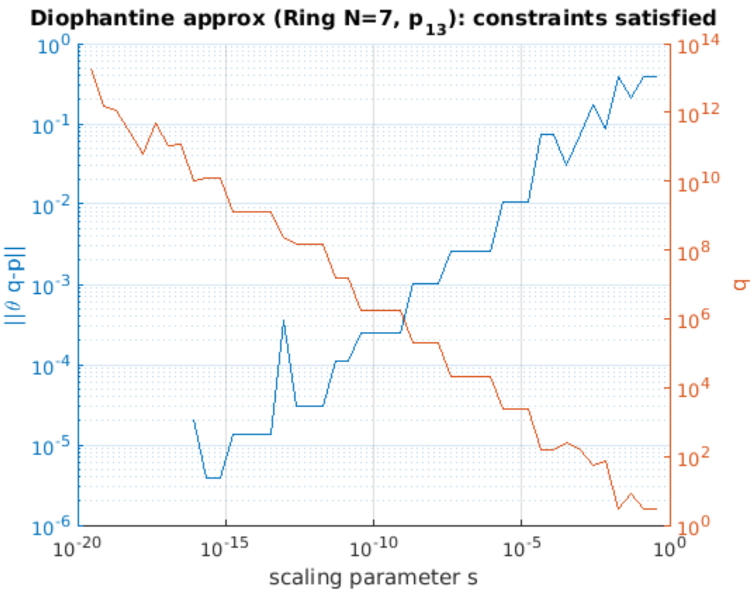
Genetic algorithm to adjust *X* in support of a conjecture

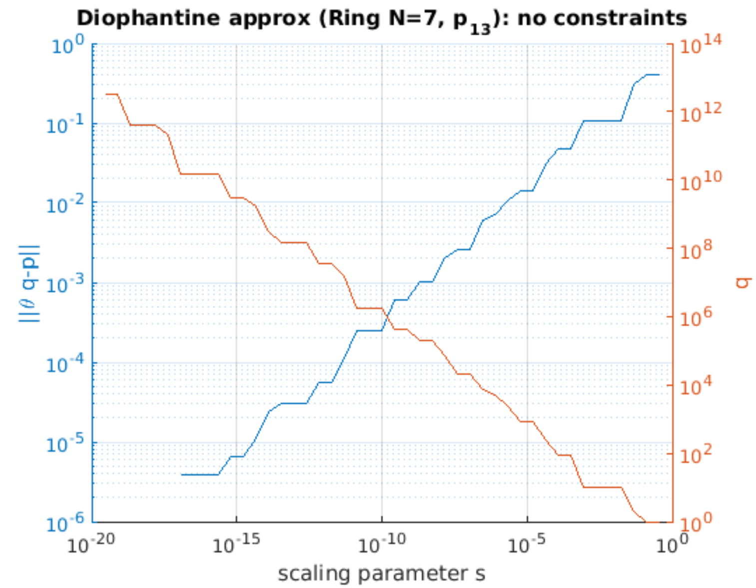
Given a user-defined s, a genetic algorithm that minimizes the number of parity violations in $p_{k,\ell}$ finds *X* in about 5 generations and a population of about 200 chromosomes.

<u>Conjecture:</u> Simultaneous Diophantine approximations of arbitrary accuracy and subject to parity constraints always exist.

<u>Proof:</u> ???

"If neither Lenstra nor Lagarias knows, then nobody knows!"
—R. Guralnick

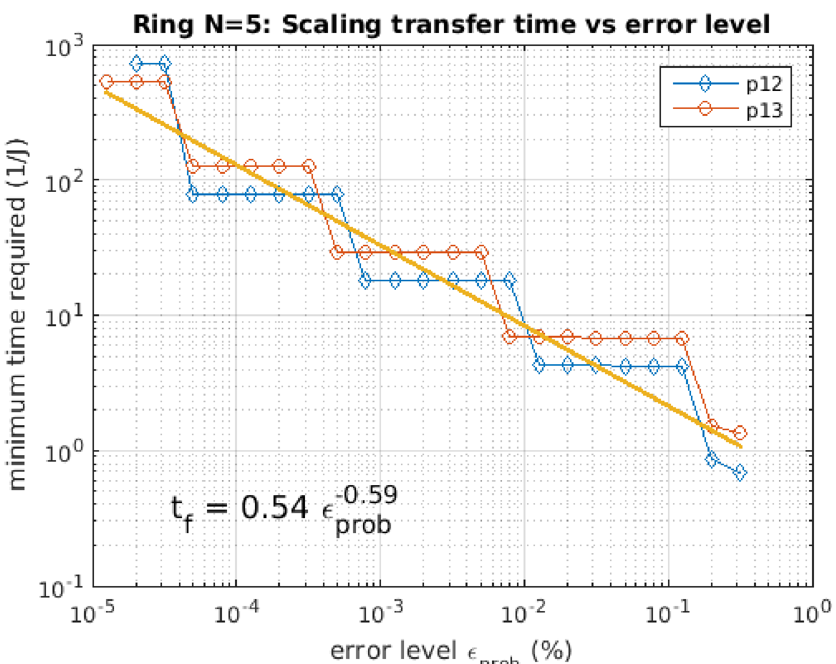


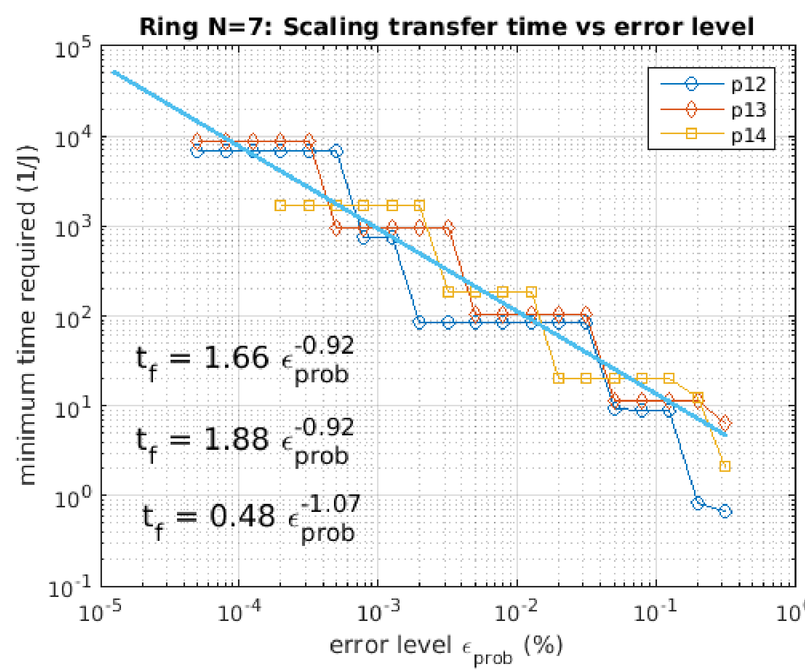


Time to reach transfer probability

$$p\left(\left|j,t\right\rangle,\left|i,0\right\rangle\right)\geq p_{\max}(i,j)-\varepsilon_{\text{prob}}$$

$$q\geq\left(\frac{\pi\bar{N}}{\sin^{-1}\left(\frac{\varepsilon_{\text{prob}}}{4K'}\right)}\right)^{\bar{N}}$$





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Gromov boundary ∂_∞

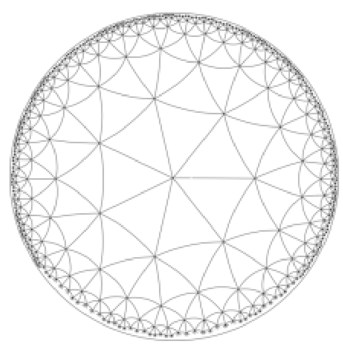
<u>Definition:</u> A ***quasi-pole*** Ω of an infinite graph G is a compact subgraph of such that there exits a geodesic ray from $\omega \in \Omega$ passing within a bounded distance B of every vertex of the graph.

<u>Definition:</u> The ***Gromov boundary*** of a graph is the equivalence class of infinite geodesic rays $r, r', r'', \ldots$ from $\omega \in \Omega$ under the relation that two rays r, r' are equivalent if their Hausdorff distance $d_H(r, r') < \infty$.

<u>Theorem</u>: The Gromov boundary is invariant under quasi-isometry.

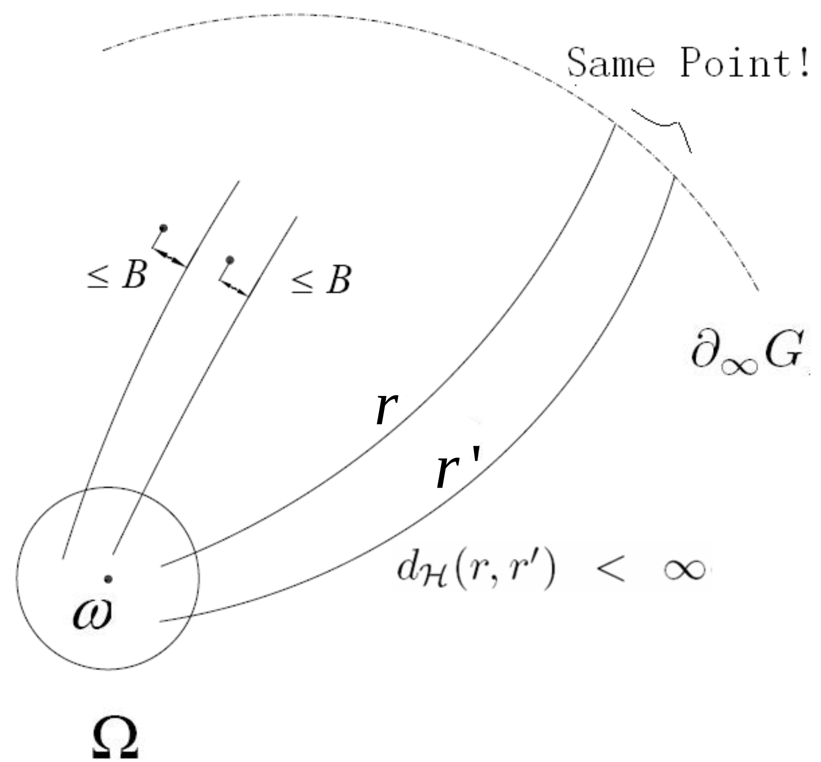
$$\partial_\infty\left(M^2_{-1}\right) = S^1$$

quasi-isometric



$$\partial_\infty\left(\text{hyperbolic tesselation of } M^2_{-1}\right) = S^1$$

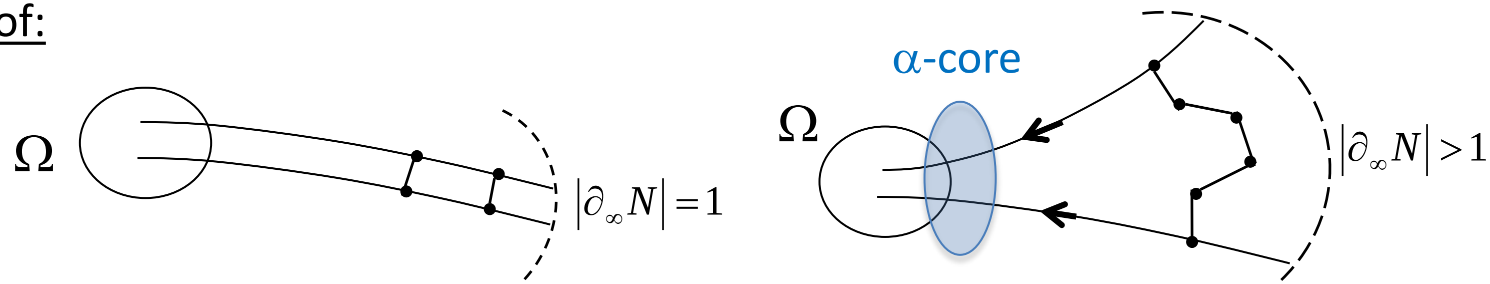
$$\partial_\infty\left(\text{binary tree}\right) = \text{Cantor set}$$



Gromov boundary and congestion

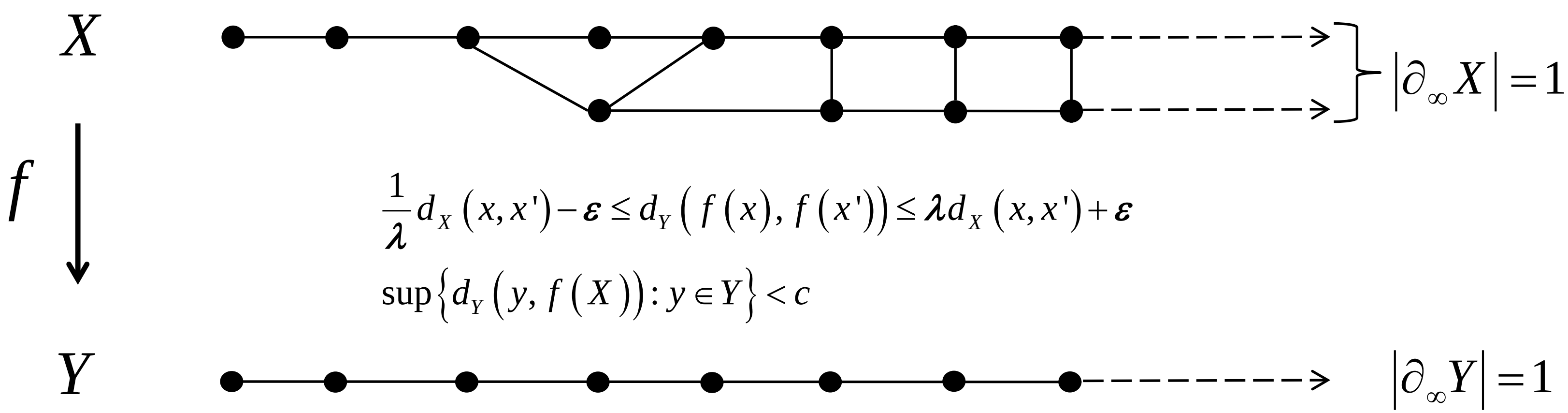
<u>Theorem (Baryshnikov):</u> Consider an infinite network (N, d_N) under a demand measure $\Lambda_\times(x,y)$ such that $xy \in E \Leftrightarrow \Lambda_\times(x,y) > 0$ and least cost path routing. If the cardinality of the Gromov boundary is 1, $|\partial_\infty N| = 1$, then there is no α-congestion core.

<u>Idea of proof:</u>



Except for *classical* networks *N* quasi-isometric to a semi-infinite chain, $|\partial_\infty N| > 1$

Example of classical network quasi-isometric to a semi-infinite chain:



$$\frac{1}{\lambda} d_X(x,x') - \varepsilon \le d_Y\left(f(x), f(x')\right) \le \lambda d_X(x,x') + \varepsilon$$

$$\sup\left\{d_Y\left(y, f(X)\right) : y \in Y\right\} < c$$

Y. Baryshnikov and G. Tucci, "Asymptotic traffic flow in a hyperbolic network," *International Symposium on Communications, Control, and Signal Processing (ISCCSP),* Rome, Italy, May 2-4, 2012.

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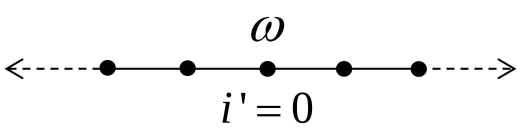
Transition probabilities in quantum chains

Basic transport equation for doubly-infinite chains:



$$p_{\max}^{1/2}(i',j') = \frac{4}{\pi^2}\left(2+\sum_{m=2,4,\ldots}\frac{4}{\left(m^2 I'^2-1\right)\left(m^2 J'^2-1\right)}\right), \quad i'+j'=0 \bmod 2$$

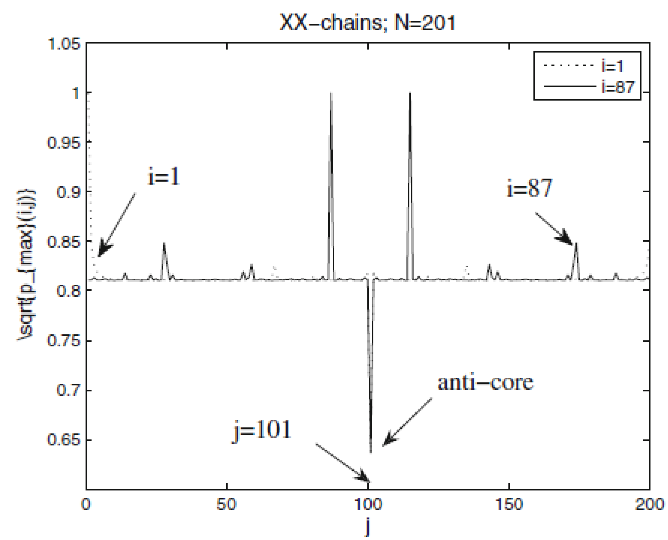
$$i'=i-\omega, \quad j'=j-\omega$$

$$I'=\frac{i'}{\gcd(i',j')}, \quad J'=\frac{j'}{\gcd(i',j')}$$

Anti-core ω defined as

$$\arg\min_j \, p_{\max}(i,j)=\omega, \quad \forall i\neq\omega,$$

$$p_{\max}(\omega,i)=0$$



Feynman path integral:

$$p_{\max}^{1/2}(i,j)=\sum_{k_1,k_2,\ldots k_{n-2}=1}^{N}\prod_{i=1}^{n-1}p_{\max}^{1/2}(k_{i-1},k_i)$$

Quantum number-theoretic computations Riemann zeta-function

Take two numbers i', j' and let $i' = \gcd(i', j')$. Then

$$p_{\max}^{1/2}(i', j') = \frac{4}{\pi^2}\left(2 + \sum_{m=2,4,\ldots} \frac{4}{\left(m^2-1\right)\left(m^2\left(\frac{j'}{\gcd(i', j')}\right)^2 - 1\right)}\right)$$

Then the $\gcd(i', j')$ is identified by

$$p_{\max}^{1/2}\left(i', j' = \gcd(i', j')\right) = 1$$

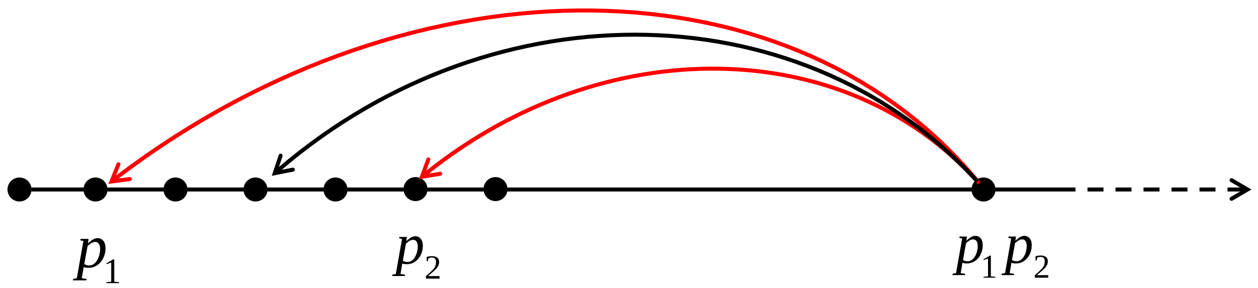
Indeed,

$$p_{\max}^{1/2}\left(i', j' = \gcd(i', j')\right) = \underbrace{\frac{4}{\pi^2}\left(2 + \sum_{m=2,4,\ldots} \frac{4}{\left(m^2-1\right)\left(m^2-1\right)}\right)}_{} = 1$$

Proof: $\zeta(2) = \frac{\pi^2}{6}$, Euler formula,
where $\zeta(s)$ is the Riemann zeta-function

Quantum number-theoretic computations

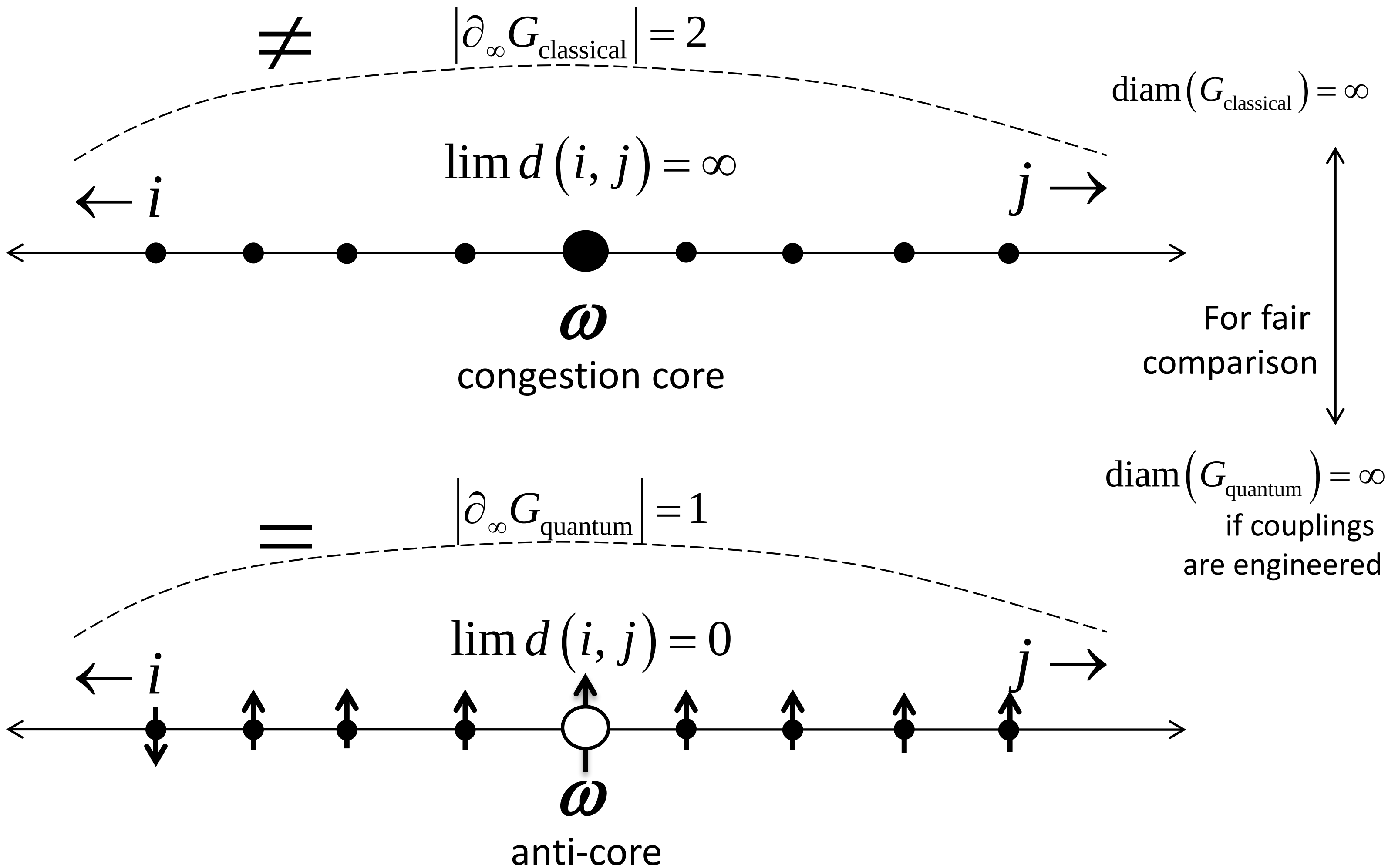
Prime number factorization



$$p_{\max}\left(i' = p_1 p_2,\, j' = p_i\right)\uparrow$$

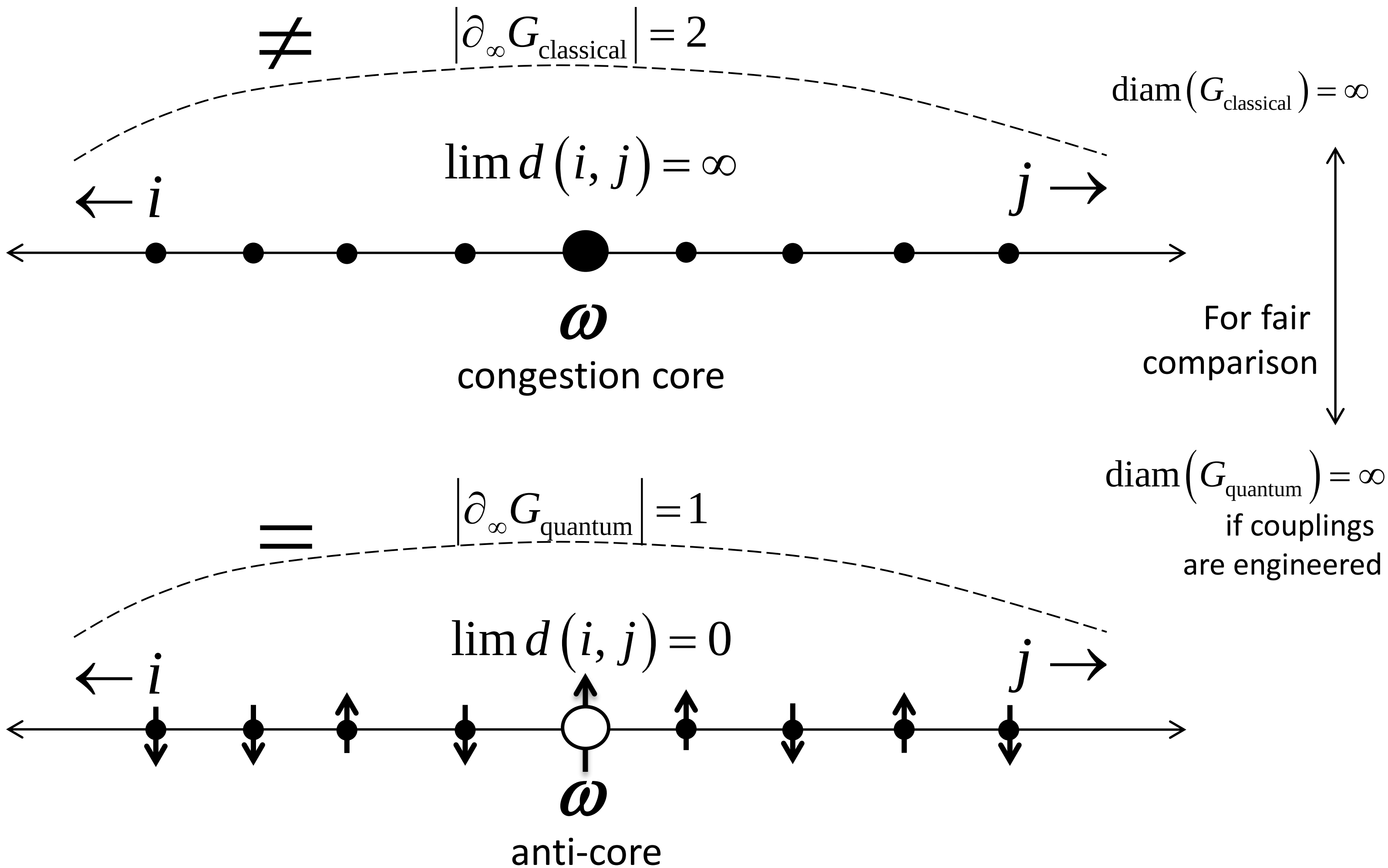
$$p_{\max}\left(i' = p_1 p_2,\, j' \neq p_i\right)\downarrow$$

Classical versus quantum networks: core versus anti-core



E. A. Jonckheere, S. Schirmer, and F. C. Langbein, ``Quantum networks: the anti-core of spin chains," *Quantum Information Processing (QINP),* volume 13, pp. 1607-1637, 2014. (DOI 10.1007/s11128-014-0755-5)

Classical versus quantum networks: core versus anti-core



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Plan of action

- Classical networks
  - Coarse geometry
  - Routing and congestion
  - Load balancing
- Quantum networks
  - Excitation-encoded information transport
  - Number-theoretic optimal transport
- Classical versus quantum networks
  - Gromov boundary: $\partial_\infty N_{\mathrm{classical}}$ versus $\partial_\infty N_{\mathrm{quantum}}$
  - Core versus anti-core
- **Conclusion & Future work**
  - Wireless networks

Conclusion & Future work

- We have developed mathematics specific to classical and quantum networks.
- On the quantum side, most work has been devoted to spintronic networks.
- Classical networks have congestion core, while quantum networks have anti-core.
- Quantum networks lead to a number-theoretic geometry.

- What mathematics need to be developed for hybrid classical-quantum networks?
- What geometric topology should be developed for quantum entanglement photonic networks?
- How to deploy surveillance at congestion core, while protecting information at anti-core?
- Could the number theoretic geometry be formalized?
- Could spintronic computers solve number theoretic problems?

Thank you!

Questions?

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