A Converse Result on Convergence Time for Opportunistic Wireless Scheduling

Michael J. Neely
University of Southern California

***Abstract*—This paper proves an impossibility result for stochastic network utility maximization for multi-user wireless systems, including multi-access and broadcast systems. Every time slot an access point observes the current channel states and opportunistically selects a vector of transmission rates. Channel state vectors are assumed to be independent and identically distributed with an unknown probability distribution. The goal is to learn to make decisions over time that maximize a concave utility function of the running time average transmission rate of each user. Recently it was shown that a stochastic Frank-Wolfe algorithm converges to utility-optimality with an error of $O(\log(T)/T)$, where T is the time the algorithm has been running. An existing $\Omega(1/T)$ converse is known. The current paper improves the converse to $\Omega(\log(T)/T)$, which matches the known achievability result. The proof uses a reduction from the opportunistic scheduling problem to a Bernoulli estimation problem. Along the way, it refines a result on Bernoulli estimation.**

***Index Terms*—optimization, stochastic processes, estimation**

I. Introduction

This paper establishes the fundamental learning rate for network utility maximization in wireless opportunistic scheduling systems, such as multiple access systems and broadcast systems. The recent work [1] shows that a stochastic Frank-Wolfe algorithm with a vanishing stepsize achieves a utility optimality gap that decays like $O(\log(T)/T)$, where T is the time the algorithm is in operation. It does this without a-priori knowledge of the channel state probabilities. This paper establishes a matching converse. A simple example system is constructed for which all algorithms have an error gap of at least $\Omega(\log(T)/T)$. Specifically, we construct a system with channel states parameterized by an unknown probability $q \in [0,1]$ such that for any algorithm, there is a set $\mathcal{Q} \subseteq [0,1]$ with measure at least $1/8$ under which the algorithm performs poorly. This is done by a novel reduction of the opportunistic scheduling problem to a problem of estimating a Bernoulli probability p from independent and identically distributed (i.i.d.) Bernoulli samples. Along the way, a refined statement regarding the regret of Bernoulli estimation is developed.

A general structure for the class of opportunistic scheduling systems is as follows: The system is assumed to operate over slotted time $t \in \{0,1,2,\ldots\}$. There are n users. Every slot $t \in \{0,1,2,\ldots\}$ an access point allocates a vector $x[t] = (x_1[t],\ldots,x_n[t])$ for transmission of independent data belonging to each user. In the case of wireless multiple access systems, the n users transmit their data over uplink channels to the access point. It is assumed they use a coordinated scheme that allows successful decoding of all transmissions at the scheduled bit rates $x[t]$. In the case of wireless broadcast systems, the access point transmits data for each user over downlink channels at the scheduled bit rates $x[t]$.

The set of all transmission rate vectors that are available on a particular slot t can change from one slot to the next. This can arise from time-varying connection properties such as channel states that vary due to device mobility. We model this time-variation by a *random state vector* $S[t] \in \mathbb{R}^m$ that is observed by the access point at the start of every slot t (where m is a positive integer that can be different from n). Assume that $\{S[t]\}_{t=0}^{\infty}$ is i.i.d. over slots with some distribution $F_S(s) = P[S[0] \leq s]$ for all $s \in \mathbb{R}^m$. The distribution function $F_S(s)$ is unknown. Define $\mathcal{D}(S[t])$ as the set of all $(x_1[t],\ldots,x_n[t])$ vectors that can be chosen on slot t when the channel state vector is $S[t]$.

The structure of $\mathcal{D}(S[t])$ depends on the network. For example, a multiple access network might allow only one user to transmit per slot. In this case we can define $S[t] = (S_1[t],\ldots,S_n[t])$ where $S_i[t]$ represents the transmission rate available to user i on slot t if that user is selected for transmission. Then $\mathcal{D}(S[t])$ is a set that contains n vectors:

$$\mathcal{D}(S[t]) = \{(S_1[t],0,0,...,0),(0,S_2[t],0,...,0), \\ ...,(0,0,...,0,S_n[t])\}$$

where the ith vector in this set corresponds to choosing user i for transmission. More sophisticated wireless signaling schemes can allow the set $\mathcal{D}(S[t])$ to contain vectors with multiple positive components. The set $\mathcal{D}(S[t])$ can be uncountably infinite in the case when transmission rates depend on an uncountably infinite set of power allocation levels that are available for scheduling.

Every slot $t \in \{0,1,2,\ldots\}$ the access point observes $S[t]$ and chooses $x[t] \in \mathcal{D}(S[t])$ in such a way that, over time, the following problem is solved:

$$\text{Maximize:} \quad \liminf_{T\to\infty} \phi\left(\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[x[t]\right]\right) \qquad (1)$$

$$\text{Subject to:} \quad x[t] \in \mathcal{D}(S[t]) \quad \forall t \in \{0,1,2,\ldots\} \qquad (2)$$

where $\phi(x_1,\ldots,x_n)$ is a given real-valued utility function of the average user transmission rates. The function ϕ is assumed to be continuous, concave, and entrywise nondecreasing. Let ϕ^* be the optimal utility value, which considers all possible algorithms that operate over an infinite time horizon, including algorithms that have perfect knowledge of the probability distribution F_S, and even including *non-causal* algorithms that

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have knowledge of future states $\{S[t]\}_{t=0}^{\infty}$.[1] It is challenging to design a (causal) scheduling algorithm that achieves utility close to ϕ^*, particularly when the distribution F_S is unknown. Algorithms that are causal (so that they have no knowledge of the future) and that have no a-priori knowledge of the distribution F_S shall be called *statistics-unaware* algorithms. This paper establishes the fundamental convergence delay required for any statistics-unaware algorithm to achieve utility that is close to the optimal value ϕ^*.

A general statistics-unaware algorithm may incorporate some type of learning or estimation of the distribution F_S or some functional of this distribution. Observations of past channel states can be exploited when making online decisions. Consider some statistics-unaware algorithm that makes decisions over time $t \in \{0, 1, 2, \ldots\}$. For each positive integer T, the expression

$$\phi\left(\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[x[t]\right]\right)$$

is the utility associated with running the algorithm over the first T slots $\{0, 1, 2, \ldots, T-1\}$. Decisions must be made intelligently at each step of the way and fast learning is crucial. How close can the achieved utility get to the optimal value ϕ^*? What time T is required?

A. Example utility functions

Different concave utility functions can be used to provide different types of performance (with corresponding fairness properties). For example, consider the linear utility

$$\phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i$$

where $a_1, \ldots, a_n$ are given nonnegative weights. Under this utility function, the problem (1)-(2) seeks to maximize a weighted sum of average transmission rates of each user. This linear utility is a trivial special case: The statistics-unaware algorithm of observing $S[t]$ at the start of every slot t and choosing $x[t] \in \mathcal{D}(S[t])$ to greedily maximize $\phi(x[t])$ can be shown to lead to immediate convergence.[2] This is because the time average expectation can be pushed inside the linear function ϕ and so maximization of immediate rewards translates into maximization of long term rewards.

This is not the case for concave but *nonlinear* utility functions. This is because the goal is to maximize a concave function of the time average, not to maximize the average of a concave function. This goal is crucial to network fairness. The greedy algorithm can be far from optimal for general concave but nonlinear utility functions. From a fairness perspective, linear utilities are undesirable. For example, suppose there are two users, at most one user can transmit per slot, and user 1 always has a strictly better channel condition than user 2 (perhaps because user 1 is closer to the access point). Maximizing the linear utility $\phi(x_1, x_2) = x_1 + x_2$ means *always* choosing user 1, so that user 2 never transmits! One way to be fair to user 2 is to change the utility function to the (nonsmooth) function

$$\phi(x_1, x_2) = \min[x_1, x_2]$$

which seeks to maximize the minimum long term transmission rate. Another common type of (smooth) utility function is

$$\phi(x_1, x_2) = \log(x_1) + \log(x_2)$$

This logarithmic utility function enforces *proportional fairness* [3] [4]. The logarithmic utility function is often modified to remove the singularity at zero:

$$\phi(x_1, x_2) = \log(1 + cx_1) + \log(1 + cx_2)$$

where $c > 0$ is a constant. Large values of c can be used to approximate the $\log(x_1) + \log(x_2)$ function. Other types of utility functions can be used for other types of fairness [5] [6] [7].

B. Prior work

The work [8] [9] develops statistics-unaware Frank-Wolfe type algorithms (with various step size rules) for solving the problem (1)-(2) for smooth utility functions using a fluid limit analysis. An alternative statistics-unaware drift-plus-penalty algorithm of [10] [2] can solve (1)-(2) for smooth or nonsmooth utility functions, and this achieves utility within ϵ of optimality with convergence time $O(1/\epsilon^2)$. Drift-plus-penalty can also be used for extended problems of multi-hop networks with power minimization and constraints [11]. Related algorithms for these extended problems are in [12] [13] [14] [15].

Recent work in [1] shows that, for smooth utility functions, a Frank-Wolfe algorithm with a constant stepsize also has convergence time $O(1/\epsilon^2)$, while a Frank-Wolfe algorithm with a vanishing stepsize yields an improved $O(\log(1/\epsilon)/\epsilon)$ convergence time. In particular, in the latter case we obtain

$$\phi\left(\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[x[t]\right]\right) \geq \phi^* - \frac{c\log(T)}{T} \quad \forall T \in \{1, 2, 3, ...\}$$

where $c > 0$ is a particular system constant. The work [1] also provides a near-matching converse of $\Omega(1/T)$. The problem of closing the logarithmic gap between the achievability bound and the converse bound was left as an open question. We resolve that open question in this paper by showing that the $O(\log(T)/T)$ gap is optimal.

A related logarithmic convergence time result is developed in [16] for the context of online convex optimization with strongly convex objective functions. The prior work [16] provides an example online convex optimization problem that immediately reduces to a problem of estimating a Bernoulli probability from i.i.d. Bernoulli samples. They then provide a deep analysis of the Bernoulli estimation problem to show, via a nested interval argument, that for any sequence of Bernoulli estimators there exists a probability $p \in [1/4, 3/4]$ under which the estimators have a sum mean square error that grows at least logarithmically in the number of samples. This prior work inspires the current paper. We show that certain opportunistic scheduling problems can also be reduced

[1]The only assumption on the algorithms considered in the definition of ϕ^* is that they are *probabilistically measurable* so they produce random vectors $x[t]$ with well defined expectations on all slots t [2].

[2]Existence of a maximizer holds under the mild additional assumption that ϕ is continuous and $\mathcal{D}(S[t])$ is a compact subset of $\mathbb{R}^n$.

to Bernoulli estimation; then we can use the Bernoulli estimation result of [16]. However, this reduction is not obvious. Online convex optimization problems have a different structure than opportunistic scheduling problems and novel reduction techniques must be used.

C. Our contributions

1) This paper proves an $\Omega(\log(T)/T)$ converse for opportunistic scheduling. This matches an existing achievability result and resolves the open question in [1] to show that this performance is optimal.
2) This paper shows that *strongly concave* utility functions cannot be used to improve the asymptotic convergence time for opportunistic scheduling problems in comparison to functions that are concave but not strongly concave. This is surprising because strong convexity/concavity provides convergence improvements in other contexts, including online convex optimization problems [17] [18] and deterministic minimization via subgradient descent [19]. This emphasizes the unique properties of opportunistic scheduling problems.
3) The technique for reducing opportunistic scheduling to Bernoulli estimation can more broadly impact future work on more complex networks (see open questions in this direction in the conclusion).
4) This paper refines the regret analysis for Bernoulli estimation theory in [16] to show that for any sequence of estimators, not only does *there exist* a probability $p \in [1/4, 3/4]$ for which the regret grows at least logarithmically, but *the set of all such values* p *has measure at least* $1/8$. This is then used for opportunistic scheduling: If any particular statistics-unaware algorithm is used, and if nature selects the channel according to a Bernoulli process with parameter p that is independently chosen over the unit interval, then with probability at least $1/8$ the algorithm will be limited by the $\Omega(\log(T)/T)$ converse bound. Shouldn't algorithms *always* be limited by this bound? No. Imagine a scheduling algorithm that makes an *a-priori guess* $\hat{q} \in [0, 1]$ about the true network probability q, and then makes decisions that are optimal under the assumption that the guess is exact. In the "lucky" situation when $\hat{q} = q$, this algorithm would perform optimally and would not be limited by the $\Omega(\log(T)/T)$ converse. Nevertheless, our analysis shows that every algorithm (including algorithms that attempt to make lucky guesses) will fail to beat the $\Omega(\log(T)/T)$ converse with probability at least $1/8$.

II. Bernoulli Estimation

This section gives preliminaries on estimating an unknown $p \in [0, 1]$ from i.i.d. Bernoulli samples $\{X_n\}_{n=1}^{\infty}$ with

$$P[X_n = 1] = p, \quad P[X_n = 0] = 1 - p$$

A. Estimation functions

Let $\{X_n^p\}_{n=1}^{\infty}$ denote a sequence of i.i.d. Bernoulli random variables with $P[X_n^p = 1] = p$ (called a *Bernoulli-p* process). The value of $p \in [0, 1]$ is unknown. On each time step n we observe the value of X_n^p and then make an estimate of p based on all observations that have been seen so far. Suppose we have some (possibly randomized) method of mapping the sequential observations into estimates of p. The goal is to produce a fundamental bound on the mean square error associated with any such sequence of estimates.

Let $\{\hat{A}_n\}_{n=1}^{\infty}$ be an infinite sequence of functions such that each function $\hat{A}_n(u, x_1, ..., x_n)$ maps a binary-valued sequence $(x_1, ..., x_n) \in \{0, 1\}^n$ and a random seed $u \in [0, 1)$ to a real number in the interval $[0, 1]$. That is, for all $n \in \{1, 2, 3, ...\}$ we have

$$\hat{A}_n : [0, 1) \times \{0, 1\}^n \to [0, 1] \tag{3}$$

The $\hat{A}_n$ functions shall be called *estimation functions*. Let U be a random variable that is uniformly distributed over $[0, 1)$ and that is independent of $\{X_n^p\}_{n=1}^{\infty}$. For $n \in \{1, 2, \ldots\}$, let A_n^p denote the estimate of p based on observations of $X_1, \ldots, X_n$:

$$A_n^p = \hat{A}_n(U, X_1^p, X_2^p, \ldots, X_n^p) \quad \forall n \in \{1, 2, 3, \ldots\} \tag{4}$$

The random variable U is used to facilitate possibly randomized decisions. The functions $\hat{A}_n$ in (3) are assumed to be *probabilistically measurable* so that A_n^p defined in (4) is a valid random variable for all $n \in \{1, 2, 3, \ldots\}$.

For each $n \in \{0, 1, 2, \ldots\}$ define $\mathbb{E}_p[(A_n^p - p)^2]$ as the mean square estimation error at time n. The expectation is with respect to the random seed U and the random Bernoulli sequence $\{X_n^p\}_{n=1}^{\infty}$ that is independent of U. The value of $\mathbb{E}_p[(A_n^p - p)^2]$ depends only on $p \in [0, 1]$, $n \in \{1, 2, 3, \ldots, \}$, and the estimation function $\hat{A}_n$ given in (3). The following theorem is from Hazan and Kale in [16].[3]

Theorem 1: (Bernoulli estimation from [16]) Fix any sequence of measurable estimation functions $\{\hat{A}_n\}_{n=1}^{\infty}$ of the form (3). There is a probability $p \in [1/4, 3/4]$ such that

$$\sum_{n=1}^{N} \mathbb{E}_p[(A_n^p - p)^2] \geq \Omega(\log(N)) \quad \forall N \in \{1, 2, 3, ...\}$$

where A_n^p is defined by (4).

B. Positive measure in the unit interval

Theorem 1 shows that for any sequence of Bernoulli estimators, there is a probability $p \in [1/4, 3/4]$ under which the estimators have sum mean square error that grows at least logarithmically in N. The next theorem shows that, not only does such a probability p exist, the set of all such probabilities p is measurable and has measure at least $1/8$ within the interval $[1/4, 3/4]$. It also generalizes to treat arbitrary powers of absolute error (including both mean square error and mean absolute error). For each $\alpha > 0$ define:

$$V_m(\alpha) = \sum_{n=1}^{m} (1/n)^{\alpha/2} \quad \forall m \in \{1, 2, 3, \ldots\}$$

[3]There is nothing special about the interval $[1/4, 3/4]$; the same result holds for any $[a, b]$ with $0 < a < b < 1$. The authors of [16] used $[1/4, 3/4]$ to compute a simple constant in front of the $\log(N)$ term. Unfortunately, a minor constant-factor error makes the constant reported in [16] too large; details and a fix of this minor issue are found in [20].

Theorem 2: [4] Fix $\alpha \in (0, 2]$. Fix any sequence of measurable estimation functions $\{\hat{A}_n\}_{n=1}^{\infty}$ of the form (3). Define $\mathcal{Q} \subseteq [1/4, 3/4]$ as the set of all $p \in [1/4, 3/4]$ such that:

$$\limsup_{m\rightarrow\infty}\left[\frac{1}{V_m(\alpha)}\sum_{n=1}^{m}\mathbb{E}_p[|A_n^p - p|^{\alpha}]\right] \geq \frac{1}{c^{\alpha}2^{3+2\alpha}}$$

where $c = \sqrt{8/3}$. Then $\mathcal{Q}$ is Lebesgue measurable with measure $\mu(\mathcal{Q}) \geq 1/8$. In particular if $\alpha = 2$ then

$$\limsup_{m\rightarrow\infty}\left[\frac{1}{\log(m+1)}\sum_{n=1}^{m}\mathbb{E}_p[|A_n^p - p|^2]\right] \geq \frac{3}{2^{10}}$$

Proof: Omitted for brevity (see technical report [20]). ■

It is important to distinguish the results of Theorems 1-2 from the Cramer-Rao estimation bound (see, for example, [21]). The Cramer-Rao bound is most conveniently applied to *unbiased estimators*. While biased versions of the Cramer-Rao bound exist, they require additional structural assumptions, such as knowledge of a (differentiable) bias function $b'(p)$ with a derivative that is bounded away from -1 so that a term $(1 + b'(p))^2$ does not vanish. Moreover, Cramer-Rao bounds are typically applied to a single estimator for time step n. In contrast, Theorems 1-2 treats the *sum* error over a *sequence* of estimators, which is essential for establishing connections to the regret of online scheduling algorithms.

III. The Converse Bound

This section constructs a simple 2-user opportunistic scheduling system with state vectors $S[t]$ described by a single probability parameter $q \in [1/4, 3/4]$. It produces a converse bound on the utility optimality gap by mapping the problem to a Bernoulli estimation problem and then using Theorem 1 and Theorem 2.

A. The example 2-user system

Consider a 2-user wireless system that operates in slotted time $t \in \{0, 1, 2, \ldots\}$. Suppose the system state is described by a sequence of i.i.d. Bernoulli variables $\{S[t]\}_{t=0}^{\infty}$ with

$$P[S[t] = 1] = q; \quad P[S[t] = 0] = 1 - q$$

where $q \in [0, 1]$ is an unknown probability. Every slot $t \in \{0, 1, 2, \ldots\}$ the system controller observes $S[t]$ and chooses a *transmission rate vector* $x[t] = (x_1[t], x_2[t])$ from a decision set $\mathcal{D}(S[t])$ given by

$$\mathcal{D}(S[t]) = \begin{cases} \{(1, 0)\} & \text{if } S[t] = 0 \\ \{(r, 1 - r^2) : r \in [0, 1]\} & \text{if } S[t] = 1 \end{cases} \quad (5)$$

In particular, if $S[t] = 0$ then the controller has no choice but to allocate $x[t] = (1, 0)$, which gives no transmission rate to user 2. On the other hand, if $S[t] = 1$ then the controller is free to allocate $x[t] = (r, 1 - r^2)$ for some $r \in [0, 1]$, which allows giving a nonzero transmission rate to user 2. Observe that under any system state and any decision, it holds that $0 \leq x_1[t] \leq 1$, $0 \leq x_2[t] \leq 1$, and $x_2[t] = 1 - x_1[t]^2$ for all t. The set of all points $(x_1[t], x_2[t]) \in \mathcal{D}(1)$ available when $S[t] = 1$ is shown as the solid curve in Fig. 1.

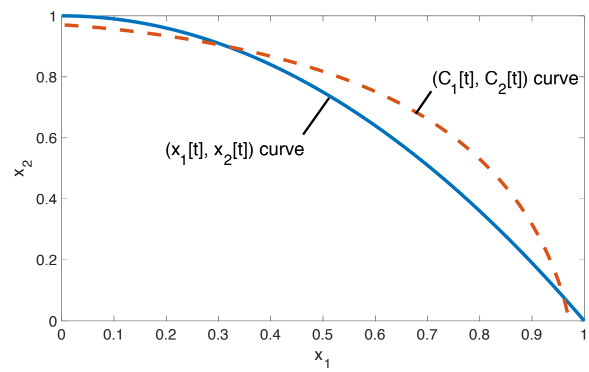

Fig. 1. The decision set $\mathcal{D}(1)$ compared to an alternative logarithmic decision set. The solid curve shows the $(x_1[t], x_2[t])$ points in $\mathcal{D}(1)$ when $S[t] = 1$; the dashed curve shows the points $(C_1[t], C_2[t])$ defined by (6)-(7).

B. Discussion of decision set $\mathcal{D}(S[t])$

This example decision set $\mathcal{D}(S[t])$ is representative of the following physical scenario: Imagine that user 2 goes offline independently every slot t with probability $1-q$ (possibly due to a time-varying channel condition, or because it allocates its resources to other tasks according to a randomized schedule). Hence, user 1 can allocate a full rate of 1 on those slots (corresponding to slots t such that $S[t] = 0$). On the other hand, during the slots in which users 1 and 2 are both online (corresponding to $S[t] = 1$), the users can simultaneously transmit but, due to interference, they cannot both transmit at the full rate of 1. On these slots, there is a tradeoff between the rates $x_1[t]$ and $x_2[t]$ that can be allocated, so that $x_2[t]$ is a nonincreasing function of $x_1[t]$. The particular nondecreasing function $x_2[t] = 1-x_1[t]^2$ that is used is shown in Fig. 1. This function is chosen for mathematical convenience (it simplifies the proof to be given).

Similar proofs can be given for related curves: For example, for slots t such that $S[t] = 1$, suppose the total bandwidth available is B and the rates of users 1 and 2 are chosen by allocating fractions of the bandwidth $\theta_1[t]$ and $\theta_2[t]$ to users 1 and 2, so that user 1 is allocated a total bandwidth of $B\theta_1[t]$, user 2 is allocated a total bandwidth of $B\theta_2[t]$, and $\theta_1[t], \theta_2[t]$ are chosen as nonnegative values that sum to 1. The users thus transmit over frequency-separated channels. Assuming each channel is an additive white Gaussian noise channel (with noise density uniform over the given frequency spectrum) and given these particular frequency division allocations on slot t, the point-to-point Shannon capacity of each channel is [21]:

$$C_1[t] = \theta_1[t]B\log\left(1 + \frac{P}{\theta_1[t]N}\right) \quad (6)$$

$$C_2[t] = \theta_2[t]B\log\left(1 + \frac{P}{\theta_2[t]N}\right) \quad (7)$$

where P and N are fixed positive parameters. The expression $\frac{P}{\theta_i[t]N}$ represents the signal-to-noise ratio for channel $i \in \{1, 2\}$ and the noise $\theta_i N$ is proportional to the bandwidth

[4] The version of this paper printed in the Infocom 2020 conference proceedings incorrectly states Theorem 2 as applying to $\alpha > 0$. This PDF corrects to $\alpha \in (0, 2]$. The case $\alpha > 2$ is not of interest because $\lim_{m\rightarrow\infty} V_m(\alpha) < \infty$ if $\alpha > 2$. The proof in technical report [20] treats $\alpha \in (0, 2]$.

used on channel i. The curve of $(C_1[t], C_2[t])$ values formed by all $(\theta_1[t], \theta_2[t])$ allocations is given in Fig. 1 (for the case $B = 0.7, P/N = 3$). This curve is different from $(r, 1-r^2)$, but it is qualitatively similar. Like the curve $(r, 1-r^2)$, it can be shown to have a *strongly concave structure*. The proof of our converse can be extended to this particular $(C_1[t], C_2[t])$ curve, and to similar curves that are strongly concave.

C. The example network utility maximization problem

For positive integers T define:

$$\overline{x}_i[T] = \frac{1}{T}\sum_{t=0}^{T-1} x_i[t] \quad \forall i \in \{1, 2\}$$

and define $\overline{x}[T] = (\overline{x}_1[T], \overline{x}_2(T))$. Let $\phi : [0, 1]^2 \to \mathbb{R}$ be a continuous and concave utility function. The goal of the network controller is to allocate $x[t]$ over time to solve

$$\text{Maximize:} \quad \liminf_{T\to\infty} \phi\left(\mathbb{E}\left[\overline{x}[T]\right]\right) \tag{8}$$

$$\text{Subject to:} \quad x[t] \in \mathcal{D}(S[t]) \quad \forall t \in \{0, 1, 2, \ldots\} \tag{9}$$

This problem indeed has the structure of (1)-(2). Define the *optimal utility* ϕ^* as the supremum value of (8) over all algorithms that satisfy (9). The specific utility function that we first consider is:

$$\phi(x_1, x_2) = \log(1 + x_1) + \log(1 + x_2)$$

This function is entrywise nondecreasing and *c-strongly concave* over the domain $(x_1, x_2) \in [0, 1]^2$ with parameter $c = 1/4$, meaning that the function $s : [0, 1]^2 \to \mathbb{R}$ given by

$$s(x_1, x_2) = \phi(x_1, x_2) + \frac{c}{2}(x_1^2 + x_2^2)$$

is concave. The main result of the paper is below.

Theorem 3: Consider the 2-user example problem (8)-(9) with utility function $\phi(x_1, x_2) = \log(1+x_1)+\log(1+x_2)$ and with an unknown parameter $q \in [0, 1]$. Under any (possibly randomized) statistics-unaware control algorithm, there is a probability $q \in [1/4, 3/4]$ such that

$$\phi(\mathbb{E}\left[\overline{x}[T]\right]) \leq \phi^* - \Omega\left(\frac{\log(T)}{T}\right) \quad \forall T \in \{1, 2, 3, \ldots\} \tag{10}$$

Furthermore, there is a measurable subset $\mathcal{Q} \subseteq [1/4, 3/4]$ with measure at least $1/8$ such that if $q \in \mathcal{Q}$ then

$$\limsup_{T\to\infty} \frac{\phi^* - \phi(\mathbb{E}\left[\overline{x}[T]\right])}{\log(T)/T} \geq \frac{3\beta^2}{2^{13}} \tag{11}$$

where $\beta = \frac{2}{3} - \frac{\sqrt{7}}{6} \approx 0.2257$.

The proof of Theorem 3 is developed in the following subsections. Observe that since the utility function $\phi(x_1, x_2) = \log(1+x_1)+\log(1+x_2)$ is smooth, the result in [1] ensures that the statistics-unaware Frank-Wolfe algorithm with vanishing stepsize can be used, without knowledge of the parameter $q \in [0, 1]$, to ensure that for all values $q \in [0, 1]$ we have

$$\phi(\mathbb{E}\left[\overline{x}[T]\right]) \geq \phi^* - O\left(\frac{\log(T)}{T}\right) \quad \forall T \in \{1, 2, 3, \ldots\}$$

In particular, the $\log(T)/T$ asymptotic converse bound of Theorem 3 can be achieved. Hence, the asymptotic $\log(T)/T$ bound is tight and the corresponding Frank-Wolfe algorithm (with vanishing stepsize) is asymptotically optimal.

D. Optimality over stationary policies

Results in [2] show that optimality for the problem (8)-(9) can be characterized by the set $\mathcal{C}$ of all one-shot expectations $\mathbb{E}\left[(x_1[0], x_2[0])\right]$ that can be achieved on slot $t = 0$.

Lemma 1: (The set $\mathcal{C}$) Fix $q \in (0, 1]$. The set $\mathcal{C}$ of all one-shot expectations $\mathbb{E}\left[(x_1[0], x_2[0])\right]$ achievable in the 2-user system is

$$\mathcal{C} = Conv(\{(1-q+qr, q(1-r^2)) \in [0, 1]^2 : r \in [0, 1]\}) \tag{12}$$

where $Conv(\cdot)$ denotes the convex hull. This set $\mathcal{C}$ is closed, bounded, convex, and is equivalently described as

$$\mathcal{C} = \Big\{(x_1, x_2) \in [0, 1]^2 : 1 - q \leq x_1 \leq 1, \\ x_1 + x_2 \geq 1, x_2 \leq 2(1 - x_1) - \frac{1}{q}(1 - x_1)^2\Big\} \tag{13}$$

Proof: We first show all points in the set $\mathcal{C}$ defined in (12) can be achieved as expectations on slot $t = 0$. Fix $r \in [0, 1]$. Consider the policy that chooses $x[0] = (1, 0)$ whenever $S[0] = 0$ and $x[t] = (r, 1 - r^2)$ whenever $S[0] = 1$. Then

$$\mathbb{E}\left[(x_1[0], x_2[0])\right] = ((1 - q) + qr, q(1 - r^2))$$

Hence, any point in $\{(1 - q + qr, q(1 - r^2)) \in [0, 1]^2 : r \in [0, 1]\}$ can be achieved as an expectation on slot 0. Points in the convex hull can be achieved by randomization. Thus, all points in $\mathcal{C}$ are achievable on slot 0. The proof that $\mathcal{C}$ is closed, bounded, convex, and equivalent to the set defined in (13) is straightforward and is omitted for brevity (see [20]).

The proof that no expectation on slot 0 can leave the set $\mathcal{C}$ defined in (13) is as follows: Consider any (possibly randomized) decision rule for choosing $(x_1[0], x_2[0]) \in \mathcal{D}(S[0])$, where $\mathcal{D}(S[0])$ is given in (5). The structure of $\mathcal{D}(S[0])$ ensures that, regardless of $S[0]$, we have $x_2[0] = 1 - x_1[0]^2$ and $0 \leq x_1[0] \leq 1$. For simplicity define the random variable $R = x_1[0]$, so that $(x_1[0], x_2[0]) = (R, 1 - R^2)$. Define constants x_1, x_2 by

$$(x_1, x_2) = (\mathbb{E}\left[R\right], \mathbb{E}\left[1 - R^2\right])$$

It suffices to show that $(x_1, x_2) \in \mathcal{C}$. Define $z = \mathbb{E}\left[R|S[0] = 1\right]$ and note that $0 \leq z \leq 1$. We have

$$\begin{aligned} x_1 &= \mathbb{E}\left[R\right] \\ &= (1 - q)\mathbb{E}\left[R|S[0] = 0\right] + q\mathbb{E}\left[R|S[0] = 1\right] \\ &= (1 - q) + qz \end{aligned} \tag{14}$$

Since $0 \leq z \leq 1$ it follows that

$$1 - q \leq x_1 \leq 1 \tag{15}$$

Furthermore, since $R \in [0, 1]$ we have $R^2 \leq R$ and so

$$x_1 + x_2 = \mathbb{E}\left[R + (1 - R^2)\right] \geq \mathbb{E}\left[R + (1 - R)\right] = 1 \tag{16}$$

Finally

$$\begin{aligned} x_2 &= \mathbb{E}\left[1 - R^2\right] \\ &= (1 - q)\mathbb{E}\left[1 - R^2|S[0] = 0\right] + q\mathbb{E}\left[1 - R^2|S[0] = 1\right] \\ &= q(1 - \mathbb{E}\left[R^2|S[0] = 1\right]) \\ &\overset{(a)}{\leq} q(1 - \mathbb{E}\left[R|S[0] = 1\right]^2) \\ &= q(1 - z^2) \end{aligned} \tag{17}$$

where (a) holds by Jensen's inequality. On the other hand, we know by (14) that $x_1 = (1-q)+qz$ and hence $z = 1-\frac{(1-x_1)}{q}$. Substituting this expression for z into the right-hand-side of (17) gives

$$x_2 \leq q - q\left(1 - \frac{(1-x_1)}{q}\right)^2 = 2(1-x_1) - \frac{(1-x_1)^2}{q} \quad (18)$$

The inequalities (15),(16),(18) imply that $(x_1, x_2) \in \mathcal{C}$ (compare with the definition of $\mathcal{C}$ given in (13)). ∎

Results in [2] imply that the optimal utility ϕ^* for problem (8)-(9) is equal to the maximum of the continuous function $\phi(x_1, x_2)$ over all (x_1, x_2) in the compact set $\mathcal{C}$:

$$\phi^* = \sup_{(x_1,x_2)\in\mathcal{C}} \phi(x_1, x_2)$$

The next lemma establishes the maximizing point (x_1^*, x_2^*) for our specific utility function. This maximizer is unique because our utility function $\phi(x_1, x_2)$ is strongly concave.

Lemma 2: (The optimal operating point) Fix $q \in (0, 1]$. Define $\phi : [0,1]^2 \to \mathbb{R}$ by $\phi(x_1, x_2) = \log(1 + x_1) + \log(1 + x_2)$. The unique maximizer of $\phi(x_1, x_2)$ over $(x_1, x_2) \in \mathcal{C}$ is a vector (x_1^*, x_2^*) that satisfies

$$(x_1^*, x_2^*) = ((1-q) + qr, q(1-r^2)) \quad (19)$$

for some particular $r \in [0, 1]$, so that $\phi(x_1^*, x_2^*) = \phi^*$. Furthermore the optimal value $r \in [0, 1]$ in (19) satisfies

$$\frac{1}{1 + x_1^*} = \frac{2r}{1 + x_2^*} \quad (20)$$

and is exactly equal to

$$r = \frac{-(2-q) + \sqrt{4q^2 - q + 4}}{3q} \quad (21)$$

The expression on the right-hand-side of (21) increases from $1/4$ to $\frac{-1+\sqrt{7}}{3}$ as q slides between 0 and 1, where $\frac{-1+\sqrt{7}}{3} \approx 0.54858$.

Proof: Considering the set $\mathcal{C}$ defined by (12), it is clear that (because $\phi(x_1, x_2)$ is entrywise nondecreasing) a maximizer occurs on the upper boundary curve $(1-q+qr, q(1-r^2))$ for $r \in [0, 1]$. The utility function associated with $r \in [0, 1]$ is

$$\log(2 - q + qr) + \log(1 + q(1 - r^2))$$

This is concave over $r \in [0, 1]$. If a point of zero derivative can be found over $r \in [0, 1]$ then that point must be optimal. Taking a derivative with respect to r and setting the result to 0 yields

$$\frac{q}{2-q+qr} + \frac{-2qr}{1+q(1-r^2)} = 0$$

Since $q > 0$, dividing by $q > 0$ and rearranging terms gives

$$\frac{1}{2-q+qr} = \frac{2r}{1+q(1-r^2)}$$

which yields (20) by the substitution $x_1^* = 1 - q + qr$, $x_2^* = q(1 - r^2)$. Rearranging the above equality again yields a quadratic equation in r that is solved by taking the only nonnegative solution, which is given in (21). It can be checked that the expression in (21) increases from $1/4$ to $\frac{-1+\sqrt{7}}{3}$ as q slides between 0 and 1. In particular, a zero-derivative point $r \in [0, 1]$ can indeed be found for all values of q that are considered. ∎

Lemma 3: (The bijection h) Define the function $h : [0, 1] \to [1/4, \frac{-1+\sqrt{7}}{3}]$ by

$$h(q) = \begin{cases} 1/4 & \text{if } q = 0 \\ \frac{-(2-q)+\sqrt{4q^2-q+4}}{3q} & \text{otherwise} \end{cases} \quad (22)$$

so $q \in (0, 1]$ implies $r = h(q)$ (where r is defined in (21)). Then h is strictly increasing, so that it has an inverse function:

$$h^{-1} : \left[1/4, \tfrac{-1+\sqrt{7}}{3}\right] \to [0, 1]$$

Further, h is continuously differentiable, satisfies $h'(q) \geq h'(1)$ for all $q \in [0, 1]$, and has the "expansion property":

$$|h(a) - h(b)| \geq \beta|a - b| \quad \forall a, b \in [0, 1] \quad (23)$$

where β is defined

$$\beta = h'(1) = \frac{2}{3} - \frac{\sqrt{7}}{6} \approx 0.2257 \quad (24)$$

Proof: Omitted for brevity (see technical report [20]). ∎

E. Statistics-unaware algorithms for utility maximization

This subsection completes the proof of Theorem 3 for the 2-user problem (8)-(9). Fix $q \in [1/4, 3/4]$ and recall that $q = P[S[t] = 1]$. Consider any *statistics-unaware* algorithm for choosing $x[t] = (x_1[t], x_2[t]) \in \mathcal{D}(S[t])$ over $t \in \{0, 1, 2, \ldots\}$, where $\mathcal{D}(S[t])$ is given in (5). In particular, the algorithm has no a-priori knowledge of q. For each $t \in \{0, 1, 2, \ldots\}$ define

$$y[t] = (y_1[t], y_2[t]) = \mathbb{E}\left[(x_1[t], x_2[t])\right]$$

Fix T as a positive integer. Define $\overline{x}[T]$ as the time average over the first T slots

$$\overline{x}[T] = \tfrac{1}{T}\textstyle\sum_{t=0}^{T-1}(x_1[t], x_2[t])$$

Taking expectations of both sides of the above equality and using the definition of $y[t]$ gives

$$\mathbb{E}\left[\overline{x}[T]\right] = \tfrac{1}{T}\textstyle\sum_{t=0}^{T-1} y[t] \quad (25)$$

Let $x^* = (x_1^*, x_2^*)$ be the optimal operating point defined in (19) of Lemma 2. Let $\phi'(x^*)^\top$ denote the gradient at x^* expressed as a row vector:

$$\begin{aligned} &\phi(x_1, x_2) = \log(1 + x_1) + \log(1 + x_2) \\ &\implies \phi'(x^*)^\top = \left[\frac{\partial\phi(x^*)}{\partial x_1}; \frac{\partial\phi(x^*)}{\partial x_2}\right] = \left[\frac{1}{1+x_1^*}; \frac{1}{1+x_2^*}\right] \end{aligned} \quad (26)$$

By concavity of ϕ we have:

$$\begin{aligned} \phi(\mathbb{E}\left[\overline{x}(T)\right]) &\overset{(a)}{\leq} \phi(x^*) + \phi'(x^*)^\top(\mathbb{E}\left[\overline{x}(T)\right] - x^*) \\ &\overset{(b)}{=} \phi(x^*) + \phi'(x^*)^\top \cdot \frac{1}{T}\sum_{t=0}^{T-1}(y[t] - x^*) \\ &\overset{(c)}{=} \phi^* + \frac{1}{T}\sum_{t=0}^{T-1}\left[\frac{y_1[t] - x_1^*}{1 + x_1^*} + \frac{y_2[t] - x_2^*}{1 + x_2^*}\right] \\ &\overset{(d)}{=} \phi^* + \frac{1}{T}\sum_{t=0}^{T-1}\left[\frac{2r(y_1[t] - x_1^*)}{1 + x_2^*} + \frac{y_2[t] - x_2^*}{1 + x_2^*}\right] \end{aligned} \quad (27)$$

where (a) holds by the gradient inequality for concave functions; (b) holds by (25); (c) holds by (26) and the fact that $\phi(x^*) = \phi^*$ for the vector $x^* = (x_1^*, x_2^*)$ defined in Lemma 2; (d) holds by (20).

Now consider a particular $t \in \{0, 1, \ldots, T-1\}$. Define $F[t]$ as the history of channel states over the first t slots:

$$F[t] = (S[0], S[1], \ldots, S[t-1])$$

with $S[-1]$ defined as a nonrandom constant so that $F[0]$ provides no information about the channel. Define $z_{F[t]}$ by

$$z_{F[t]} = \mathbb{E}\left[x_1[t] | S[t] = 1, F[t]\right] \tag{28}$$

By Jensen's inequality we have

$$z_{F[t]}^2 \leq \mathbb{E}\left[x_1[t]^2 | S[t] = 1, F[t]\right] \tag{29}$$

We have

$$\begin{aligned}
&\mathbb{E}\left[(x_1[t], x_2[t]) | F[t]\right] \\
&\overset{(a)}{=} \mathbb{E}\left[(x_1[t], x_2[t]) | S[t] = 0, F[t]\right](1-q) \\
&\quad + \mathbb{E}\left[(x_1[t], x_2[t]) | S[t] = 1, F[t]\right] q \\
&\overset{(b)}{=} (1, 0)(1-q) + \mathbb{E}\left[(x_1[t], 1 - x_1[t]^2) | S[t] = 1, F[t]\right] q \\
&\overset{(c)}{\leq} (1, 0)(1-q) + (z_{F[t]}, 1 - z_{F[t]}^2) q \\
&= (1 - q + q z_{F[t]}, q - q z_{F[t]}^2)
\end{aligned} \tag{30}$$

where (a) holds by conditioning on the events $\{S[t] = 1\}$ and $\{S[t] = 0\}$ (which are independent of $F[t]$); (b) holds by the decision set structure $\mathcal{D}(S[t])$ in (5); (c) holds with entrywise inequality by (28) and (29). Taking expectations of (30) with respect to the random $F[t]$ and using the law of iterated expectations gives (using $\mathbb{E}\left[x_i[t]\right] = y_i[t]$):

$$(y_1[t], y_2[t]) \leq (1 - q + q\mathbb{E}\left[z_{F[t]}\right], q - q\mathbb{E}\left[z_{F[t]}^2\right]) \tag{31}$$

On the other hand, recall from Lemma 2 that

$$(x_1^*, x_2^*) = (1 - q + qr, q(1 - r^2)) \tag{32}$$

Using (31) and (32) together gives

$$\begin{aligned}
&2r(y_1[t] - x_1^*) + (y_2[t] - x_2^*) \\
&\leq 2rq\left(\mathbb{E}\left[z_{F[t]}\right] - r\right) + q\left(r^2 - \mathbb{E}\left[z_{F[t]}^2\right]\right) \\
&= -q\mathbb{E}\left[(z_{F[t]} - r)^2\right] \\
&\leq -q\mathbb{E}\left[\left(\left[z_{F[t]}\right]_{h(0)}^{h(1)} - r\right)^2\right]
\end{aligned} \tag{33}$$

where $[z_{F[t]}]_{h(0)}^{h(1)}$ projects $z_{F[t]}$ onto the interval $[h(0), h(1)]$ and the final inequality holds because we know $r \in [h(0), h(1)]$, and the distance between $z_{F[t]}$ and r must be greater than or equal to their distances when projected onto the interval $[h(0), h(1)]$. Now we know that $h : [0, 1] \to [h(0), h(1)]$ is bijective and so we can define

$$\theta[t] = h^{-1}\left(\left[z_{F[t]}\right]_{h(0)}^{h(1)}\right) \tag{34}$$

so that $[z_{F[t]}]_{h(0)}^{h(1)} = h(\theta[t])$. Substituting this and $r = h(q)$ into (33) gives

$$2r(y_1[t] - x_1^*) + (y_2[t] - x_2^*) \leq -q\mathbb{E}\left[(h(\theta[t]) - h(q))^2\right]$$

Substituting the above inequality into (27) yields:

$$\begin{aligned}
\phi(\mathbb{E}\left[\overline{x}(T)\right]) &\leq \phi^* - \frac{q}{T(1 + x_2^*)} \sum_{t=0}^{T-1} \mathbb{E}\left[(h(\theta[t]) - h(q))^2\right] \\
&\overset{(a)}{\leq} \phi^* - \frac{1}{8T} \sum_{t=0}^{T-1} \mathbb{E}\left[(h(\theta[t]) - h(q))^2\right] \\
&\overset{(b)}{\leq} \phi^* - \frac{1}{8T} \sum_{t=1}^{T-1} \mathbb{E}\left[(h(\theta[t]) - h(q))^2\right]
\end{aligned} \tag{35}$$

where (a) holds by the fact that $x_2^* \in [0, 1]$ and $q \in [1/4, 3/4]$ so that $q/(1 + x_2^*) \geq 1/8$; (b) holds by neglecting the nonnegative term for $t = 0$. By the expansion property of h in (23) we obtain

$$\boxed{\phi(\mathbb{E}\left[\overline{x}(T)\right]) \leq \phi^* - \frac{\beta^2}{8T} \sum_{t=1}^{T-1} \mathbb{E}\left[(\theta[t] - q)^2\right]} \tag{36}$$

where $\beta = h'(1) \approx 0.2257$ is defined in Lemma 3.

Here is the crucial observation: For each slot $t \in \{1, 2, 3, \ldots\}$ we can view $\theta[t]$ as defined in (34) as a deterministic estimator of q based on the t observations $\{S[0], S[1], \ldots, S[t-1]\}$.

Indeed, $z_{F[t]}$ defined in (28) is a deterministic function of $F[t] = (S[0], \ldots, S[t-1])$, and $\theta[t]$ as defined in (34) is determined by first projecting $z_{F[t]}$ to the interval $[h(0), h(1)]$ and then mapping the result through the continuous deterministic function $h^{-1} : [h(0), h(1)] \to [0, 1]$. In particular, $\theta[t] \in [0, 1]$. With this observation, we get from Theorem 1 that there exists a value $q \in [1/4, 3/4]$ such that:

$$\sum_{t=1}^{T-1} \mathbb{E}\left[(\theta[t] - q)^2\right] \geq \Omega(\log(T)) \quad \forall T \in \{1, 2, 3, \ldots\}$$

Substituting this into (36) gives

$$\phi(\mathbb{E}\left[\overline{x}[T]\right]) \leq \phi^* - \Omega(\log(T)/T)$$

This establishes the inequality (10) of Theorem 3.

Finally, assuming $T \geq 2$ and rearranging (36) gives

$$\frac{T\phi^* - T\phi(\mathbb{E}\left[\overline{x}[T]\right])}{\log(T)} \geq \frac{\beta^2}{8\log(T)} \sum_{t=1}^{T-1} \mathbb{E}\left[(\theta[t] - q)^2\right] \tag{37}$$

Again observing that $\{\theta[t]\}_{t=1}^{\infty}$ is a sequence of deterministic estimators of q, from Theorem 2 we know there is a set $\mathcal{Q} \subseteq [1/4, 3/4]$ with measure $\mu(\mathcal{Q}) \geq 1/8$ such that for all $q \in \mathcal{Q}$ we have

$$\limsup_{T\to\infty} \frac{1}{\log(T)} \sum_{t=1}^{T-1} \mathbb{E}\left[(\theta[t] - q)^2\right] \geq \frac{3}{2^{10}}$$

Substituting this into (37) yields

$$\limsup_{T\to\infty} \frac{T\phi^* - T\phi(\mathbb{E}\left[\overline{x}[T]\right])}{\log(T)} \geq \frac{\beta^2}{8} \frac{3}{2^{10}}$$

This completes the proof of Theorem 3. □

F. Discussion

The $O(\log(T)/T)$ achievability result derived in [1] holds for smooth and concave utility functions and does not require strong concavity. The $\Omega(\log(T)/T)$ converse bound of Theorem 3 was carried out using a smooth and *strongly concave* utility function. This was intentional: *This shows that, for these opportunistic scheduling problems, strong concavity cannot improve the fundamental convergence time.* This is surprising because strong convexity/concavity is known to significantly improve convergence time in other optimization scenarios, including deterministic subgradient minimization [19] and online convex programming [18] [17].

G. Extension to other utility functions

Replace the utility function $\log(1+x_1)+\log(1+x_2)$ with the more general function $\phi : [0,1]^2 \to \mathbb{R}$:

$$\phi(x_1, x_2) = \phi_1(x_1) + \phi_2(x_2)$$

where $\phi_1(x)$ and $\phi_2(x)$ are concave and strictly increasing over $[0,1]$. The converse proof can be repeated with mild additional assumptions on ϕ_1 and ϕ_2. The main idea is to use the *implicit function theorem* of real analysis to show existence of a strictly increasing and continuously differentiable function $h : (0,1) \to (0,1)$ (different from the h function given for the log utility function in (22)) such that for each $q \in (0,1)$, the value $h(q)$ is the r value needed to define the optimal operating point $(x_1^*, x_2^*) \in \mathcal{C}$ associated with this new utility:

$$(x_1^*, x_2^*) = ((1-q) + qr, q(1-r^2))$$

It must also be shown that there is a $\beta > 0$ such that $h'(q) \geq \beta$ for all $q \in [1/4, 3/4]$ so that the proof can proceed from (35) to (36). These properties are established in the next lemma. They allow the $\Omega(\log(T)/T)$ converse proof to be repeated using the modified estimator:

$$\theta[t] = h^{-1}\left([\mathbb{E}\left[x_1[t] | S[t]=1, F[t]\right]]_{h(1/4)}^{h(3/4)} \right)$$

Lemma 4: (General utilities) Suppose $\phi_1(x)$ and $\phi_2(x)$ are twice continuously differentiable functions that satisfy:

- Assumption 1: $\phi_1'(x) > 0$ and $\phi_2'(x) > 0 \quad \forall x \in [0,1]$.
- Assumption 2: $\phi_1''(x) < 0$ and $\phi_2''(x) < 0 \quad \forall x \in [0,1]$.
- Assumption 3: $\phi_1'(1) < 2\phi_2'(0)$.[5]

Then for each $q \in (0,1)$ the equation:

$$\phi_1'(1-q+qr) + \phi_2'(q(1-r^2))(-2r) = 0 \tag{38}$$

has a unique solution $r \in (0,1)$. Further, there is a continuously differentiable function $h : (0,1) \to (0,1)$ with the property that for each $q \in (0,1)$, $h(q)$ is the unique value of $r \in (0,1)$ for which (q,r) satisfies (38). Also, there is a $\beta > 0$ such that $h'(q) \geq \beta$ for all $q \in [1/4, 3/4]$.

Proof: Define $g(q,r)$ for $q \in (0,1)$ and $r \in [0,1]$ by

$$g(q,r) = \phi_1'(1-q+qr) + \phi_2'(q(1-r^2))(-2r) \tag{39}$$

[5]Note that Assumption 3 is implied by Assumption 2 in the special case $\phi_1(x) = \phi_2(x)$ for all $x \in [0,1]$.

Then $g(q,r) = 0$ if and only if (38) holds. Fix $q \in (0,1)$. Since ϕ_1 and ϕ_2 are twice differentiable, $g(q,r)$ is a continuous function of r. We have

$$g(q,0) \overset{(a)}{=} \phi_1'(1-q) > 0$$
$$g(q,1) \overset{(b)}{=} \phi_1'(1) - 2\phi_2'(0) < 0$$

where (a) holds by Assumption 1 and (b) by Assumption 3. By the intermediate value theorem, there must exist a value $r \in (0,1)$ such that $g(q,r) = 0$. To show *uniqueness* of this value $r \in (0,1)$, it suffices to show that $g(q,r)$ is strictly decreasing in r:

$$\begin{aligned} \frac{\partial g(q,r)}{\partial r} &= \underbrace{\phi_1''(1-q+qr)q}_{<0} + \underbrace{\phi_2''(q(1-r^2))q(2r)^2}_{<0} \\ &\quad + \underbrace{(-2)\phi_2'(q(1-r^2))}_{<0} < 0 \end{aligned} \tag{40}$$

where the underbrace inequalities hold by Assumptions 1, 2, and the inequalities $0 < q < 1$, $0 < r < 1$. Thus, uniqueness holds, and for each $q \in (0,1)$ we can define $h(q)$ as the unique value in $(0,1)$ such that $g(q, h(q)) = 0$. Since $g(q,r)$ is continuously differentiable (with respect to both q and r) over the open set $(q,r) \in (0,1) \times (0,1)$, and since $\partial g / \partial r \neq 0$, the *implicit function theorem* of real analysis can be applied to conclude $h(q)$ is continuously differentiable.

It remains to prove existence of $\beta > 0$ such that that $h'(q) \geq \beta$ for all $q \in [1/4, 3/4]$. We have $g(q, h(q)) = 0$ for all $q \in (0,1)$ and so by (39):

$$\phi_1'(1-q+qh(q)) - 2h(q)\phi_2'(q(1-h(q)^2)) = 0 \quad \forall q \in (0,1)$$

Taking a derivative with respect to q gives

$$\begin{aligned} 0 &= \phi_1''(1-q+qh(q))(-1+h(q)+qh'(q)) \\ &\quad - 2h(q)\phi_2''(q(1-h(q)^2))(1-h(q)^2 - 2qh(q)h'(q)) \\ &\quad - 2h'(q)\phi_2'(q(1-h(q)^2)) \end{aligned} \tag{41}$$

Suppose there is a $q \in (0,1)$ for which $h'(q) \leq 0$. Then by Assumption 1, the last term in (41) satisfies

$$2h'(q)\phi_2'(q(1-h(q)^2)) \leq 0$$

and so by (41):

$$\begin{aligned} 0 \geq{} & \underbrace{\phi_1''(1-q+qh(q))}_{<0}\underbrace{(-1+h(q)+qh'(q))}_{<0} \\ & \underbrace{-2h(q)\phi_2''(q(1-h(q)^2))}_{>0}\underbrace{(1-h(q)^2 - 2qh(q)h'(q))}_{>0} \end{aligned}$$

So $0 > 0$, a contradiction. Thus, $h'(q)$ is a continuous and always-positive function over the interval $(0,1)$. Therefore, it has a strictly positive minimum value over the compact interval $[1/4, 3/4]$. Call this minimum β. ∎

IV. Conclusion

This paper establishes a converse bound of $\Omega(\log(T)/T)$ on the utility gap for opportunistic scheduling. This matches a recently established achievability bound of $O(\log(T)/T)$. This means that $\log(T)/T$ is the optimal asymptotic behavior.

The bound in this paper was proven for an example 2-user system with a strongly concave utility function. This demonstrates the surprising result that strong concavity of the utility function cannot improve the asymptotic convergence time for opportunistic scheduling systems. This is in contrast to other optimization scenarios, such as online convex optimization, where strong convexity/concavity is known to significantly improve asymptotic convergence. The converse proof constructed a nontrivial mapping of the opportunistic scheduling problem to a Bernoulli estimation problem and used a prior result on the regret associated with Bernoulli estimation. The paper also develops a refinement on Bernoulli estimation to show that for any sequence of Bernoulli estimators, not only do probabilities exist for which the estimators perform poorly, but such probabilities have measure at least $1/8$ in the unit interval. This is used to show that for any opportunistic scheduling algorithm, if nature chooses a Bernoulli state distribution by selecting the Bernoulli probability uniformly over the unit interval, the algorithm is limited by the $\Omega(\log(T)/T)$ bound with probability at least $1/8$.

The converse bound of this paper was established for a simple 2-user system. This means that *there exist* systems that are limited by the $\Omega(\log(T)/T)$ bound. The techniques in this paper link opportunistic scheduling to estimation problems and can likely be used in future work to investigate bounds on more general networks, including networks with state variables $S[t]$ that are described by more complex distributions. This motivates the following open questions: Can refined bounds be established for non-Bernoulli $S[t]$ processes? Can more detailed coefficients of the $\log(T)/T$ curve be obtained in terms of simple parameters of the distribution on $S[t]$? The Cramer-Rao bound of estimation theory allows bounds for non-Bernoulli variables that depend on *Fisher information* of the underlying probability distribution. However, it is currently unclear how to reduce a general opportunistic scheduling problem to a generalized (non-Bernoulli) estimation problem, and it is not clear how to incorporate Fisher information concepts to provide "regret" type bounds for networks.

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