Cantor's Diagonal Argument for Different Levels of Infinity

Michael J. Neely
University of Southern California
http://www-bcf.usc.edu/~mjneely

Abstract

These notes develop the classic Cantor diagonal argument to show that some infinite sets are so large that their elements cannot be described as an infinite list. The fact that there are an infinite number of levels of infinity is also discussed, together with a related paradox.

A. *Cardinality*

A set is *finite* if it has a finite number of elements. Specifically, a finite set has a number of elements equal to a non-negative integer (the set with no elements is called the *empty set*). For example, the following set of numbers is finite because it has only three elements:

$$\{3.4, 2.7, 9\}$$

Another example is the following set of three colors:

$$\{\text{red}, \text{blue}, \text{green}\}$$

The sets $\{3.4, 2.7, 9\}$ and $\{\text{red}, \text{blue}, \text{green}\}$ are composed of very different things. However, they are both finite. Further, they both have the same "size" because they can be put into a *one-to-one correspondence* where every element of the first set is matched to a unique element of the second, and all elements of the second set are included in this assignment of matches:

$$\begin{aligned} 3.4 &\leftrightarrow \text{red} \\ 2.7 &\leftrightarrow \text{blue} \\ 9 &\leftrightarrow \text{green} \end{aligned}$$

The above is just one of several different ways to construct a one-to-one correspondence between the sets $\{3.4, 2.7, 9\}$ and $\{\text{red}, \text{blue}, \text{green}\}$. For example, one could swap the color assignments of 3.4 and 2.7 to obtain the following alternative one-to-one correspondence:

$$\begin{aligned} 3.4 &\leftrightarrow \text{blue} \\ 2.7 &\leftrightarrow \text{red} \\ 9 &\leftrightarrow \text{green} \end{aligned}$$

It is not possible to create a one-to-one correspondence between the 2-element set $\{6, 8.9\}$ and the 3-element set {red, blue, green}, because any attempt would necessarily leave out a color.

Definition 1: Two sets are said to have the *same cardinality* if there exists a one-to-one correspondence between them.

Therefore, the sets $\{3.4, 2.7, 9\}$ and $\{\text{red}, \text{blue}, \text{green}\}$ have the same cardinality, but the sets $\{6, 8.9\}$ and {red, blue, green} do not. The word "cardinality" in the world of sets can be interpreted as "size." Two finite sets have the same cardinality if and only if they have the same number of elements.

B. *Infinite and countably infinite sets*

A set is *infinite* if it is not finite. That is, an infinite set is one that has an infinite number of elements. An example of an infinite set is the set of all positive integers:

$$\{1, 2, 3, \ldots\}$$

Another example is the set of all even positive integers:

$$\{2, 4, 6, \ldots\}$$

The set of positive integers includes all even positive integers, but also includes more (in particular, it includes all odd positive integers). However, according to Definition 1, the two sets have the same cardinality because there exists a one-to-one correspondence between them:

$$\begin{array}{ccc} 1 & \leftrightarrow & 2 \\ 2 & \leftrightarrow & 4 \\ 3 & \leftrightarrow & 6 \\ 4 & \leftrightarrow & 8 \\ & \cdots & \end{array}$$

so that each positive integer n is assigned to the even positive integer $2n$.

Definition 2: An infinite set is said to be *countably infinite* if it has the same cardinality as the set of positive integers.

According to the above definition, the set of even integers is a countably infinite set. Now consider any infinite set $\mathcal{X}$ whose elements can be written as an infinite list $\{x_1, x_2, x_3, \ldots\}$. It is assumed here that the list contains each element of $\mathcal{X}$ once and only once. Then $\mathcal{X}$ is a countably infinite set, as shown by the following one-to-one correspondence:

$$\begin{array}{ccc} 1 & \leftrightarrow & x_1 \\ 2 & \leftrightarrow & x_2 \\ 3 & \leftrightarrow & x_3 \\ 4 & \leftrightarrow & x_4 \\ & \cdots & \end{array}$$

so that each positive integer n is assigned to the element x_n.

It follows that a set is countably infinite if and only if its elements can be written out as an infinite list, where each element is on the list once and only once. For this reason, countably infinite sets are often called *listable sets*. One may wonder if *all* infinite sets are countably infinite. Can we take any infinite set and write all of its elements out in an infinite list? The surprising answer is "no." The remainder of these notes show that the set of all real numbers between 0 and 1 is a set that is so large that its elements cannot be listed.

C. *Decimal expansion of a real number*

Let $[0, 1)$ denote the set of all real numbers that are greater than or equal to 0 and strictly less than 1. Every real number in $[0, 1)$ can be written using a decimal expansion. For example:

$$\begin{aligned} 0.348 &= 3 \times 10^{-1} + 4 \times 10^{-2} + 8 \times 10^{-3} \\ 0.4294 &= 4 \times 10^{-1} + 2 \times 10^{-2} + 9 \times 10^{-3} + 4 \times 10^{-4} \\ 0.703 &= 7 \times 10^{-1} + 0 \times 10^{-2} + 3 \times 10^{-3} \end{aligned}$$

A number might have a decimal expansion with an infinite number of digits:

$$\begin{aligned} 0.3333\ldots &= 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3} + 3 \times 10^{-4} + \cdots \\ 0.212121\ldots &= 2 \times 10^{-1} + 1 \times 10^{-2} + 2 \times 10^{-3} + 1 \times 10^{-4} + 2 \times 10^{-5} + 1 \times 10^{-6} \cdots \end{aligned}$$

The decimal expansion of a number is unique, *except for the possibility of having an infinite tail of 9s*. The following examples show that, if we are allowed to use an infinite tail of 9s, we can often write the same number in two different ways:

$$\begin{aligned} 0.999999\ldots &= 1 \\ 0.1399999\ldots &= 0.14 \end{aligned}$$

This fact about infinite tails of 9s can be formally proven, but that is a tangential detail that is omitted for brevity.[1] In the special case when a number can be written with two different decimal expansions, one of those expansions must have an infinite tail of 9s while the other does not. In that case, we can just choose the expansion that does *not* have the infinite tail of 9s. This is formalized by the following fact.

Fact 1: Every real number x in the set $[0, 1)$ can be uniquely written as a decimal expansion:

$$x = 0.a_1a_2a_3\cdots = a_1 \times 10^{-1} + a_2 \times 10^{-2} + a_3 \times 10^{-3} + \cdots$$

where the digits $a_1, a_2, a_3, \ldots$ satisfy the following:

- Each digit a_i is an integer in the 10-element set $\{0, 1, 2, \ldots, 9\}$.
- The sequence of digits $\{a_1, a_2, a_3, \ldots\}$ does not have an infinite tail of 9s.

D. *Cantor's diagonal argument*

Definition 3: A set is *uncountably infinite* if it is infinite but not countably infinite.

This subsection proves the existence of an uncountably infinite set. In particular, it proves that the set of all real numbers in the interval $[0, 1)$ is uncountably infinite. The proof starts by assuming that $[0, 1)$ is countably infinite, and then reaches a contradiction.[2]

Suppose the set $[0, 1)$ is countably infinite, so that its elements can be written as an infinite list (we will reach a contradiction). Then we can list all elements of $[0, 1)$ as $\{x_1, x_2, x_3, \ldots\}$ where:

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}a_{14}a_{15}\cdots \\ x_2 &= 0.a_{21}a_{22}a_{23}a_{24}a_{25}\cdots \\ x_3 &= 0.a_{31}a_{32}a_{33}a_{34}a_{35}\cdots \\ &\cdots \\ x_i &= 0.a_{i1}a_{i2}a_{i3}a_{i4}a_{i5}\cdots \\ &\cdots \end{aligned}$$

where each number x_i on the list is written with its unique decimal expansion. Now draw a box around each "diagonal digit" in this list of decimal expansions:

$$\begin{aligned} x_1 &= 0.\boxed{a_{11}}a_{12}a_{13}a_{14}a_{15}\cdots \\ x_2 &= 0.a_{21}\boxed{a_{22}}a_{23}a_{24}a_{25}\cdots \\ x_3 &= 0.a_{31}a_{32}\boxed{a_{33}}a_{34}a_{35}\cdots \end{aligned}$$

The idea is to construct a new real number x^* that is not on the list by designing its decimal expansion $0.b_1b_2b_3\cdots$ such that each digit b_i differs from the boxed digit a_{ii}. Specifically, define the real number x^* as follows:

$$x^* = 0.b_1b_2b_3b_4b_5\cdots$$

where each digit b_i is defined:

$$b_i = \begin{cases} 8 & \text{if } a_{ii} = 7 \\ 7 & \text{if } a_{ii} \neq 7 \end{cases}$$

[1]A proof uses the fact that, for all integers k, one has the identity $\sum_{i=k}^{\infty}[9 \times 10^{-i}] = 10^{-k+1}$.

[2]Formally, one should first note that $[0, 1)$ is indeed an infinite set. For example, it includes all numbers in the infinite list $\{1/2, 1/3, 1/4, 1/5, \ldots\}$.

Since the decimal expansion of x^* has only 7s and 8s, it has no infinite tail of 9s. Thus, $0.b_1b_2b_3\cdots$ is the unique decimal expansion of x^*. Also, by this construction we have:

$$\begin{array}{ccc} b_1 & \neq & a_{11} \\ b_2 & \neq & a_{22} \\ b_3 & \neq & a_{33} \\ & \cdots & \end{array}$$

so that, for every digit i, we have $b_i \neq a_{ii}$. *It follows that x^* is a real number that is not on the list, since its unique decimal expansion $0.b_1b_2b_3b_4\cdots$ differs (in at least one digit) from the unique decimal expansion of every number on the list.*

For example, x^* cannot be the same as x_3 because:

$$\begin{array}{cccccccccc} x^* & = & 0. & b_1 & b_2 & \boxed{b_3} & b_4 & b_5 & \cdots \\ x_3 & = & 0. & a_{31} & a_{32} & \boxed{a_{33}} & a_{34} & a_{35} & \cdots \end{array}$$

and $b_3 \neq a_{33}$. Likewise, x^* cannot be the same as x_4 because:

$$\begin{array}{cccccccccc} x^* & = & 0. & b_1 & b_2 & b_3 & \boxed{b_4} & b_5 & \cdots \\ x_3 & = & 0. & a_{41} & a_{42} & a_{43} & \boxed{a_{44}} & a_{45} & \cdots \end{array}$$

and $b_4 \neq a_{44}$.

However, x^* is indeed a real number in the interval $[0, 1)$ because its decimal expansion has only 7s and 8s, so:

$$0 \leq 0.7777777\ldots \leq x^* \leq 0.888888\ldots < 1$$

Therefore, x^* is a real number in the interval $[0, 1)$ that is not on the list. This contradicts the assumption that the list contains all real numbers in $[0, 1)$.

E. Discussion

The proof in the previous subsection assumed existence of an infinite list $\{x_1, x_2, x_3, \ldots\}$ that contains all real numbers in the interval $[0, 1)$, and reached a contradiction by finding a particular real number x^* in $[0, 1)$ that was *not* on the list. One might object as follows: Why don't we simply take the old list and add to it the single real number x^*? This would form a new list $\{x^*, x_1, x_2, x_3, \ldots\}$, where x^* is now the first on the list, the old x_1 is now the second on the list, and so on.

This does not work because the argument can be repeated to find yet another number y^* in $[0, 1)$ that is not on the infinite list $\{x^*, x_1, x_2, x_3, \ldots\}$. Adding y^* to the list to form yet another list $\{y^*, x^*, x_1, x_2, x_3, \ldots\}$ does not help, for the same reason. Overall, the proof required us to only find one such number that was not on the list. However, it turns out that for any list containing numbers in $[0, 1)$, the set of all real numbers in $[0, 1)$ that are *not* on the list is, in fact, uncountably infinite.

F. Diagonalization exercises

Exercise 1: Let $\{x_1, x_2, x_3, ...\}$ be a sequence of real numbers in the interval $[0, 1)$. For each positive integer i, the unique decimal expansion of x_i is given by $x_i = 0.a_{i1}a_{i2}a_{i3}...$, so the expansion does not have an infinite tail of 9s, and $x_i = \sum_{j=1}^{\infty} a_{ij}10^{-j}$. Construct a real number $y \in [0, 1)$ that is not on the list $\{x_1, x_2, x_3, ...\}$.

Exercise 2: Let $\{x_1, x_2, x_3, \ldots\}$ be a listing of all rational numbers in the set $[0, 1)$. For each positive integer i, define the decimal expansion of x_i as:

$$x_i = 0.a_{i1}a_{i2}a_{i3}\cdots = \sum_{k=1}^{\infty} a_{ik}10^{-k}$$

where the expansion does not have an infinite tail of 9s. We want to prove the following number z is irrational:

$$z = \sum_{k=1}^{\infty} a_{kk}10^{-k}$$

a) For each positive integer k, define:

$$b_k = \begin{cases} 3 & \text{if } a_{kk} = 5 \\ 5 & \text{if } a_{kk} \neq 5 \end{cases}$$

Define $y = \sum_{k=1}^{\infty} b_k 10^{-k}$. Prove that y is irrational.

b) Use part (a) to prove that z is irrational. *Hint: A number is rational if and only if its decimal expansion has an eventually repeating pattern. Suppose $\{a_{11}, a_{22}, a_{33}, \ldots\}$ has an eventually repeating pattern.*

G. Infinite levels of infinity

A finite set $\mathcal{A}$ has *strictly smaller cardinality* than another finite set $\mathcal{B}$ if the first has fewer elements. It is possible to precisely define the notion of strictly smaller cardinality for arbitrary (possibly infinite) sets. The statement "set $\mathcal{A}$ has strictly smaller cardinality than set $\mathcal{B}$" is often written with the simpler shorthand:

$$|\mathcal{A}| < |\mathcal{B}|$$

where the notation $|\mathcal{A}|$ is read "the cardinality of $\mathcal{A}$."

For any set $\mathcal{A}$, one can define a new set $P(\mathcal{A})$, called the *power set of* $\mathcal{A}$, as the set of all subsets of $\mathcal{A}$. For example, the 3-element set $\mathcal{A} = \{1, 2, 3\}$ has an 8-element power set:

$$P(\mathcal{A}) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

where "$\{\}$" represents the "empty set." Another simple argument of Cantor shows that any non-empty set $\mathcal{A}$ has strictly smaller cardinality than its power set. That is:

$$|\mathcal{A}| < |P(\mathcal{A})| \tag{1}$$

This can be used to show that there are an infinite number of levels of infinity. Indeed, let $\mathbb{N}$ denote the set of all positive integers (also called *natural numbers*). Then:

$$|\mathbb{N}| < |P(\mathbb{N})| < |P(P(\mathbb{N}))| < |P(P(P(\mathbb{N})))| < \cdots \tag{2}$$

Thus, starting with the infinite set $\mathbb{N}$, the power set operation iteratively produces new infinite sets with strictly larger cardinality than all of their predecessors.

H. A deep paradox

The chain of sets given in (2) shows there are infinitely many levels of infinity. That is, there are infinitely many different infinite cardinalities. Of course, this iterative method can only demonstrate existence of a *countably infinite* number of levels of infinity. This leads one to wonder if the set of all distinct levels of infinity is in fact a countably infinite set.

There is a simple proof that the answer is "no." It is a resounding "no." It is the loudest possible "no" that one can imagine! *The set of all distinct levels of infinity is so large that it cannot even be called a set!* The standard proof of this fact starts by assuming we can represent all distinct infinite cardinalities as a set, call it set $\mathcal{C}$. It then uses a variation on the power set principle to construct a new infinite set with cardinality that is not in $\mathcal{C}$.

The specific proof is as follows: Suppose $\mathcal{C}$ is the collection of all distinct infinite cardinalities. Suppose that for each distinct cardinality c in $\mathcal{C}$, we have a representative set $\mathcal{X}_c$ that has this cardinality. Hence, we have a collection of sets, one for each possible cardinality. It can be shown that the cardinality of any particular set in the collection is less than or equal to the cardinality of the *union of all sets* in the collection, and so the power set of this union has *strictly greater* cardinality than any set in the collection.[3] That is, the following holds for every cardinality d in the set $\mathcal{C}$:

$$|\mathcal{X}_d| \leq |\cup_{c \in \mathcal{C}} \mathcal{X}_c| < |P(\cup_{c \in \mathcal{C}} \mathcal{X}_c)| \tag{3}$$

This result is arguably one of the deepest "paradoxes" of mathematics.

[3]Formally, the following intuitive fact can be proven: if $\mathcal{A}$ and $\mathcal{B}$ are sets that satisfy $\mathcal{A} \subseteq \mathcal{B}$ (so that all elements of $\mathcal{A}$ are also in $\mathcal{B}$), then $|\mathcal{A}| \leq |\mathcal{B}|$. The first inequality in (3) follows immediately from this fact together with the observation that $\mathcal{X}_d \subseteq \cup_{c \in \mathcal{C}} \mathcal{X}_c$ whenever d is in the set $\mathcal{C}$. The second inequality in (3) follows from (1). The notation "$\cup_{c \in \mathcal{C}} \mathcal{X}_c$" means "the union of all sets $\mathcal{X}_c$ such that c is in the set $\mathcal{C}$" and represents the set of all elements that are in one or more of the sets in the collection.