Abstract

These notes review basic calculus, including the chain rule, product rule, and the fundamental theorems of calculus. The rules for integration by substitution and integration by parts are derived as consequences. Finally, worked examples that apply the substitution rule and the integration by parts rule are given. The functions \( f(x) \) considered in these notes are assumed to be real valued functions defined over \( x \in \mathbb{R} \), that is, \( f : \mathbb{R} \to \mathbb{R} \).

I. BASIC CALCULUS RESULTS

A. Differentiation

Let \( f(x), g(x) \) be differentiable functions and let \( f'(x), g'(x) \) be the derivatives.

- Chain rule: If \( h(x) = f(g(x)) \) then \( h'(x) = f'(g(x))g'(x) \).

- Product rule: If \( h(x) = f(x)g(x) \) then \( h'(x) = f'(x)g(x) + f(x)g'(x) \).

B. Fundamental theorems of calculus

- \( \int_a^b f'(x)dx = f(b) - f(a) \). [Assuming \( f'(x) \) is a continuous function]

- \( \frac{d}{dx} \left[ \int_a^x f(t)dt \right] = f(x) \). [Assuming \( f(t) \) is a continuous function]

II. INTEGRATION TECHNIQUES

A. Substitution rule for integration

If \( f(u), g(x), g'(x) \) are continuous functions and \( a, b \) are real numbers, then

\[
\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du
\]

Proof: (Application of chain rule for differentiation) Define \( h(x) = F(g(x)) \), where \( F(x) \) is an antiderivative of \( f(x) \). By the chain rule we get:

\[ h'(x) = f(g(x))g'(x) \]

Integrating both sides gives:

\[ \int_a^b h'(x)dx = \int_a^b f(g(x))g'(x)dx \]

On the other hand, we have:

\[
\int_a^b h'(x) \overset{(a)}{=} h(b) - h(a) \overset{(b)}{=} F(g(b)) - F(g(a)) \overset{(c)}{=} \int_{g(a)}^{g(b)} f(u)du
\]

where (a) holds by the fundamental theorem of calculus applied to the continuous function \( h'(x) \), (b) holds by definition of \( h \), and (c) holds by the fundamental theorem of calculus applied to the continuous function \( f(u) \).
B. Integration by parts

If \( f(x), f'(x), g(x), g'(x) \) are continuous functions and \( a, b \) are real numbers, then

\[
\int_a^b f(x)g'(x)dx = f(x)g(x) \bigg|_a^b - \int_a^b f'(x)g(x)dx
\]

where \( f(x)g(x) \big|_a^b = f(b)g(b) - f(a)g(a) \).

Proof: (Application of product rule for differentiation) Define \( h(x) = f(x)g(x) \). The product rule gives:

\[ h'(x) = f'(x)g(x) + f(x)g'(x) \]

Integrating both sides gives:

\[ \int_a^b h'(x)dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx \]

On the other hand,

\[ \int_a^b h'(x)dx \overset{(a)}{=} h(b) - h(a) \overset{(b)}{=} f(b)g(b) - f(a)g(a) \]

where (a) holds by the fundamental theorem of calculus applied to the continuous function \( h'(x) \), and (b) holds by definition of \( h \).

C. Examples of integration via the substitution rule

Consider the equation for the substitution rule:

\[
\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du
\]

The substitution rule can be implemented by performing the change of variables

\[
\begin{align*}
    u &= g(x) \\
    du &= g'(x)dx
\end{align*}
\]

and then remembering to appropriately change the integration limits.

1) Solving \( \int_a^b xe^{x^2}dx \): Define \( u = x^2 \), \( du = 2xdx \). Then:

\[
\int_a^b xe^{x^2}dx = \int_a^b \frac{1}{2} e^u du = \frac{1}{2} e^{b^2} - \frac{1}{2} e^{a^2}
\]

2) Solving \( \int_1^3 e^x \cos(e^x)dx \): Define \( u = e^x \), \( du = e^x dx \). Then:

\[
\int_1^3 e^x \cos(e^x)dx = \int_e^3 \cos(u)du = \sin(3) - \sin(e)
\]

D. Examples of integration by parts

Consider the integration by parts equation:

\[
\int_a^b f(x)g'(x)dx = f(x)g(x) \bigg|_a^b - \int_a^b g(x)f'(x)dx
\]

Integration by parts can be implemented by the change of variables:

\[
\begin{align*}
    u &= f(x) & dv &= g'(x)dx \\
    du &= f'(x)dx & v &= g(x)
\end{align*}
\]
1) Solving $\int_1^5 x \cos(x)dx$: Define $u = x$, $dv = \cos(x)dx$. Then:

\[ u = x, \quad dv = \cos(x)dx \]
\[ du = dx, \quad v = \sin(x) \]

So:

\[ \int_1^5 x \cos(x)dx = x\sin(x)|_1^5 - \int_1^5 \sin(x)dx \]
\[ = x\sin(x)|_1^5 + \cos(x)|_1^5 \]
\[ = 5\sin(5) + \cos(5) - \sin(1) - \cos(1) \]

2) Solving $\int_0^\infty xe^{-x}dx$: Define $u = x$, $dv = e^{-x}dx$. Then:

\[ u = x, \quad dv = e^{-x}dx \]
\[ du = dx, \quad v = -e^{-x} \]

So:

\[ \int_0^\infty xe^{-x}dx = -xe^{-x}|_0^\infty - \int_0^\infty (e^{-x})dx = 0 + \int_0^\infty e^{-x}dx = -e^{-x}|_0^\infty = 1 \]