

Transient response of velocity fluctuations in inertialess channel flows of viscoelastic fluids

Mihailo R. Jovanović and Satish Kumar

Abstract—We examine transient responses of velocity fluctuations in inertialess channel flows of viscoelastic fluids. Such fluids have broad applications in modern technology including the design and control of polymer processing operations and the development of strategies to efficiently mix fluids in microfluidic devices. For streamwise-constant three-dimensional fluctuations, we demonstrate analytically the existence of initial conditions that lead to quadratic scaling of the kinetic energy density with the Weissenberg number, We . This illustrates that in strongly elastic channel flows of viscoelastic fluids, velocity fluctuations can exhibit significant transient growth even in the absence of inertia. Furthermore, we show that the fluctuations in streamwise velocity achieve $\mathcal{O}(We)$ growth over a time scale $\mathcal{O}(We)$ before eventual asymptotic decay. We also demonstrate that the large transient responses originate from the stretching of polymer stress fluctuations by a background shear and draw parallels between streamwise-constant inertial flows of Newtonian fluids and streamwise-constant inertialess flows of viscoelastic fluids.

Index Terms—Elastic turbulence, inertialess flows, microfluidic mixing, polymers, transient response, viscoelastic fluids.

I. INTRODUCTION

In contrast to Newtonian fluids (e.g. air and water) which transition to turbulence under the influence of inertia, viscoelastic fluids may become turbulent even at low Reynolds numbers (Re) due to additional dynamics associated with the polymeric contribution to the stress tensor [1]–[4]. Transition to turbulence in these fluids is not only of fundamental scientific importance, but is also relevant to applications. Examples include the design and control of polymer processing operations and the development of strategies to efficiently mix fluids in microfluidic devices [5]–[8].

By now it is well understood that standard linear stability analysis is misleading when it comes to predicting the early stages of transition in wall-bounded shear flows of Newtonian fluids [9]. The non-normal nature of the dynamical generator in the linear stability problem allows for disturbances to the linearized Navier-Stokes (NS) equations to be significantly amplified, indicating that these equations have high sensitivity (i.e., they are exceedingly sensitive to external disturbances) and small robustness margins (i.e., they are exceedingly sensitive to small changes in the underlying model) [10]–[13]. Thus, background disturbances and imperfections in the laboratory represent particular examples

of *modeling uncertainty* that may conspire to yield experimental observations which are at odds with results from standard linear stability analysis.

For wall-bounded shear flows of viscoelastic fluids, the dynamical generator in the linear stability problem is also non-normal. This has led a number of researchers to apply the tools of nonmodal stability theory in order to better understand transition in these flows. These analyses have primarily used the Oldroyd-B constitutive equation, the simplest model for a dilute solution of polymer molecules [6]. Both analytical and numerical studies of transient growth in viscoelastic fluids have been carried out [14]–[18], and these show that initial conditions exist which can grow significantly at short times before decaying at long times. The transient growth can occur when fluid inertia is much weaker than fluid elasticity, and even when inertia is completely absent. Except for Ref. [18], the disturbances considered are two-dimensional (2D) and in the plane of the base flow. In Ref. [18], three-dimensional (3D) disturbances are studied using the upper convected Maxwell model (a special case of the Oldroyd-B model), but these involve perturbations only to the stresses and not to the velocity field.

In this paper we analyze transient responses of velocity fluctuation in channel flows of viscoelastic fluids. In our analysis we set the Reynolds number to zero, which yields a static-in-time relationship between velocity and polymer stress fields (i.e., inertialess flow of a polymeric fluid). The motivation for considering inertialess flow comes from the observation that polymeric fluids can become turbulent even at very small Re [1], [2]. The dynamics of the polymer stress tensor are represented using the Oldroyd-B constitutive equation. Although velocity and polymer stress fluctuations are fully 3D in general, we focus here on the case where the fluctuations are 3D but streamwise-constant (i.e., the streamwise wavenumber k_x is set to zero); our prior work on stochastically driven flows shows that such perturbations are most amplified by the linearized dynamics [19], [20]. Additionally, considerably more analytical progress can be made for this case compared to the case of 2D (streamwise-varying but spanwise-constant) fluctuations. In particular, several explicit scaling relationships are developed and numerically stable results are obtained even at large Weissenberg number, We , which is the ratio of the fluid relaxation time to the characteristic flow time.

For streamwise-constant 3D fluctuations, we show that velocity fluctuations can exhibit significant transient growth even in inertialess flows. In particular, we analytically establish that the fluctuations in streamwise velocity achieve $\mathcal{O}(We)$ growth over a time scale $\mathcal{O}(We)$ before eventual asymptotic decay. Furthermore, we identify the stretching of polymer stress fluctuations by a background shear as the culprit behind this large transient growth and provide a com-

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parison of streamwise-constant inertial flows of Newtonian fluids and inertialess flows of Oldroyd-B fluids.

II. PROBLEM SETUP

A. Governing equations and base state

We consider an incompressible flow of a polymeric fluid in a straight 3D channel (Fig. 1). The fluid density is given by ρ , λ is the polymer relaxation time, and η_s , η_p are the solvent and polymeric contributions to the shear viscosity, respectively. By scaling length with the channel half height L , velocity with the largest base velocity U_o , time with λ , pressure with $(\eta_s + \eta_p)U_o/L$, and polymer stresses with $\eta_p U_o/L$, the equations of motion, continuity, and the polymeric contribution to the stress tensor can be written as

$$Re\dot{\mathbf{V}} = We(\beta\Delta\mathbf{V} + (1-\beta)\nabla\cdot\mathbf{T} - \nabla P - Re\mathbf{V}\cdot\nabla\mathbf{V}), \quad (1a)$$

$$0 = \nabla\cdot\mathbf{V}, \quad (1b)$$

$$\dot{\mathbf{T}} = We(\mathbf{T}\cdot\nabla\mathbf{V} + (\mathbf{T}\cdot\nabla\mathbf{V})^T - \mathbf{V}\cdot\nabla\mathbf{T}) + \nabla\mathbf{V} + (\nabla\mathbf{V})^T - \mathbf{T}. \quad (1c)$$

Here, a dot identifies a partial derivative with respect to time t , \mathbf{V} is the velocity vector, P is the pressure, \mathbf{T} is the polymer stress tensor, ∇ is the gradient, and $\Delta = \nabla\cdot\nabla$ is the Laplacian. Eqs. (1) contain three parameters: the Reynolds number, $Re = \rho U_o L / (\eta_s + \eta_p)$, characterizes the ratio of inertial to viscous forces; the Weissenberg number, $We = \lambda U_o / L$, captures the product of the polymer relaxation time λ and the typical velocity gradient U_o/L ; and the viscosity ratio, $\beta = \eta_s / (\eta_s + \eta_p)$, quantifies the contribution of the solvent to the total viscosity. The constitutive equation (Eq. (1c)) is given for an Oldroyd-B fluid. This equation describes history-dependent elastic deformation and is obtained from kinetic theory by representing each polymer molecule by an infinitely extensible Hookean spring connecting two spherical beads [6].

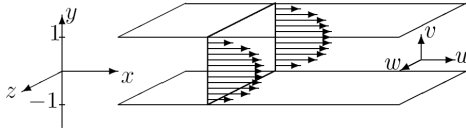


Fig. 1. Schematic of a three-dimensional channel flow.

In shear-driven (Couette) and pressure-driven (Poiseuille) channel flows, Eqs. (1) exhibit the following steady-state solutions for base velocity, $\bar{\mathbf{v}}$, and base polymer stress, $\bar{\boldsymbol{\tau}}$,

$$\bar{\mathbf{v}} = \begin{bmatrix} U(y) \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\boldsymbol{\tau}} = \begin{bmatrix} 2We(U'(y))^2 & U'(y) & 0 \\ U'(y) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where y denotes the wall-normal coordinate, $U(y) = y$ in Couette flow, $U(y) = 1 - y^2$ in Poiseuille flow, and $U'(y) = dU(y)/dy$.

In the limit of vanishing inertial forces, i.e. for $Re = 0$, one obtains inertialess flow of an Oldroyd-B fluid for which Eq. (1a) simplifies to the following static-in-time equation

$$0 = \beta\Delta\mathbf{V} + (1-\beta)\nabla\cdot\mathbf{T} - \nabla P. \quad (2)$$

Clearly, for $\beta = 1$ Eq. (2) decouples from Eq. (1c) and inertialess flow of a Newtonian fluid is recovered.

B. Streamwise-constant linearized model

We confine our study to streamwise-constant 3D fluctuations in inertialess flows of an Oldroyd-B fluid. This signifies that the dynamics evolve in the (y, z) -plane, and that the flow fluctuations in all three spatial directions are considered; for example, $\mathbf{v} = \mathbf{v}(y, z, t) = [u \ v \ w]^T$, where u , v , and w , respectively, denote the streamwise, wall-normal, and spanwise velocity fluctuations. This particular model lends itself to an explicit characterization of the transient growth dependence on the Weissenberg number, as we will show below. We will utilize this explicit We -scaling to draw parallels between streamwise-constant inertial flows of Newtonian fluids and inertialess flows of Oldroyd-B fluids.

The linearized dynamics can be obtained by decomposing each field in Eqs. (2), (1b), and (1c) into the sum of base and fluctuating parts (e.g., $\mathbf{T} = \bar{\boldsymbol{\tau}} + \boldsymbol{\tau}$), and by neglecting quadratic terms in flow fluctuations in Eq. (1c) (see Appendix A). For streamwise-constant flows, the (y, z) -plane streamfunction ψ can be introduced to rewrite v and w as $\{v = \partial_z \psi, w = -\partial_y \psi\}$, which implies that velocity fluctuations automatically satisfy the continuity equation. Furthermore, the pressure can be eliminated from Eq. (2) to express ψ and u in terms of the polymer stress fluctuation tensor $\boldsymbol{\tau}$. For purely harmonic fluctuations in the z -direction, application of this procedure yields

$$\begin{aligned} \beta\Delta^2\psi &= -(1-\beta)(ik_z\partial_y\tau_{22} - (\partial_{yy} + k_z^2)\tau_{23} - ik_z\partial_y\tau_{33}), \\ \beta\Delta u &= -(1-\beta)(\partial_y\tau_{12} + ik_z\tau_{13}), \end{aligned}$$

where the same notation is used to represent the field (e.g., $\psi(y, z, t)$) and its spanwise Fourier transform (e.g., $\psi(y, k_z, t)$). Here, τ_{ij} with $i, j = \{1, 2, 3\}$ denotes the ij th component of the polymer stress fluctuation tensor $\boldsymbol{\tau}$, k_z is the spanwise wavenumber, i is the imaginary unit, $\Delta = \partial_{yy} - k_z^2$ with homogeneous Dirichlet boundary conditions, and $\Delta^2 = \partial_{yyy} - 2k_z^2\partial_{yy} + k_z^4$ with homogeneous Cauchy (both Dirichlet and Neumann) boundary conditions. Thus, the streamfunction (and consequently v and w) at each time instant depends only on the current value of $\boldsymbol{\tau}_1 = [\tau_{22} \ \tau_{23} \ \tau_{33}]^T$; similarly, the streamwise velocity u is instantaneously equilibrated with the gradient of $\boldsymbol{\tau}_2 = [\tau_{12} \ \tau_{13}]^T$. To highlight this dependence, we write

$$\psi = \mathbf{C}_\psi \boldsymbol{\tau}_1, \quad u = \mathbf{C}_u \boldsymbol{\tau}_2, \quad (3)$$

where operators \mathbf{C}_ψ and \mathbf{C}_u are given by

$$\begin{aligned} \mathbf{C}_\psi &= -\frac{1-\beta}{\beta} \Delta^{-2} \begin{bmatrix} ik_z\partial_y & -(\partial_{yy} + k_z^2) & -ik_z\partial_y \end{bmatrix}, \\ \mathbf{C}_u &= -\frac{1-\beta}{\beta} \Delta^{-1} \begin{bmatrix} \partial_y & ik_z \end{bmatrix}. \end{aligned} \quad (4)$$

We further rearrange the six independent components of the polymer stress tensor $\boldsymbol{\tau}$ into the vector $[\boldsymbol{\tau}_1^T \ \boldsymbol{\tau}_2^T \ \tau_3]^T$, $\tau_3 = \tau_{11}$, and bring the linearization of Eq. (1c) to the following form

$$\dot{\boldsymbol{\tau}}_1 = -\boldsymbol{\tau}_1 + \mathbf{F}_{1\psi}\psi, \quad (5a)$$

$$\dot{\boldsymbol{\tau}}_2 = We(\mathbf{F}_{21}\boldsymbol{\tau}_1 + \mathbf{F}_{2\psi}\psi) + (-\boldsymbol{\tau}_2 + \mathbf{F}_{2u}u), \quad (5b)$$

$$\dot{\tau}_3 = We^2\mathbf{F}_{3\psi}\psi + We(\mathbf{F}_{32}\boldsymbol{\tau}_2 + \mathbf{F}_{3u}u) - \tau_3, \quad (5c)$$

where the \mathbf{F} -operators are given by Eq. (13) in Appendix A. A careful examination of the linearized version of constitutive equation (1c) shows that, from a physical point of view, $\mathbf{F}_{1\psi}$ and \mathbf{F}_{2u} produce gradients of velocity fluctuations (i.e., $\nabla \mathbf{v}$), $\mathbf{F}_{2\psi}$ captures both transport and stretching of base polymer stress by velocity fluctuations (i.e., $\mathbf{v} \cdot \nabla \bar{\boldsymbol{\tau}}$ and $\bar{\boldsymbol{\tau}} \cdot \nabla \mathbf{v}$), and \mathbf{F}_{21} and \mathbf{F}_{32} represent stretching of polymer stress fluctuations by base shear (i.e., $\boldsymbol{\tau} \cdot \nabla \bar{\mathbf{v}}$). Furthermore, operators $\mathbf{F}_{3\psi}$ and \mathbf{F}_{3u} in Eq. (5c) quantify transport and stretching of a base polymer stress by velocity fluctuations (i.e., $\mathbf{v} \cdot \nabla \bar{\boldsymbol{\tau}}$ and $\bar{\boldsymbol{\tau}} \cdot \nabla \mathbf{v}$), respectively.

Substitution of Eq. (3) into Eqs. (5) suggests a one-way coupling from Eq. (5a) to Eq. (5b) and from Eqs. (5a) and (5b) to Eq. (5c),

$$\dot{\boldsymbol{\tau}}_1 = \mathbf{A}_{11} \boldsymbol{\tau}_1, \quad (6a)$$

$$\dot{\boldsymbol{\tau}}_2 = We \mathbf{A}_{21} \boldsymbol{\tau}_1 + \mathbf{A}_{22} \boldsymbol{\tau}_2, \quad (6b)$$

$$\dot{\boldsymbol{\tau}}_3 = We^2 \mathbf{A}_{31} \boldsymbol{\tau}_1 + We \mathbf{A}_{32} \boldsymbol{\tau}_2 - \boldsymbol{\tau}_3, \quad (6c)$$

where the We -independent operators \mathbf{A} are given by Eq. (14) in Appendix A. Thus, for streamwise-constant fluctuations, we conclude that: (i) the dynamics of $\boldsymbol{\tau}_1$ is not influenced by the other polymer stress fluctuations; (ii) the evolution of $\boldsymbol{\tau}_2$ depends on the evolution of $\boldsymbol{\tau}_1$; and (iii) there is no coupling from $\boldsymbol{\tau}_3 = \boldsymbol{\tau}_{11}$ to the other polymer stress components in Eqs. (6). In particular, this demonstrates that in streamwise-constant inertialess flows of Oldroyd-B fluids, evolution of $\boldsymbol{\tau}_{11}$ does not influence evolution of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$. Since the velocity fluctuation vector \mathbf{v} only depends on $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, it follows that evolution of $\boldsymbol{\tau}_{11}$ is inconsequential to the dynamics of u , v , and w .

III. TRANSIENT RESPONSE OF VELOCITY FLUCTUATIONS

The main objective of this paper is to show that velocity fluctuations in viscoelastic fluids can experience significant transient growth even in the absence of inertia. This necessitates study of the temporal evolution of the fluctuations' kinetic energy density. In this section, we examine transient growth of this measure of the size of velocity fluctuations as a function of the Weissenberg number. We establish that the presence of initial conditions in $\boldsymbol{\tau}_1$ leads to $\mathcal{O}(We)$ responses of the streamwise velocity fluctuation. In contrast, the responses from all other initial conditions to all other velocity components are We -independent. Since the L_2 norm of velocity fluctuations determines kinetic energy, this shows that initial conditions leading to quadratic scaling of the energy density of the streamwise velocity fluctuation with We can be configured. Therefore, in strongly elastic flows of Oldroyd-B fluids the streamwise velocity can achieve significant transient growth even in the absence of inertia if the initial configuration of the polymers is such that $\boldsymbol{\tau}_1(0) \neq 0$. We also demonstrate that large transient responses arise from stretching of polymers by background shear and provide a comparison of streamwise-constant inertial flows of Newtonian fluids and inertialess flows of Oldroyd-B fluids. In particular, we show that, at the level of velocity fluctuation dynamics, polymer stretching and the Weissenberg number in elasticity-dominated flows of viscoelastic fluids effectively assume the role of vortex tilting and the Reynolds number in inertia-dominated flows of Newtonian fluids.

A. Kinetic energy density

At any spanwise wavenumber k_z and time t , the kinetic energy density of velocity fluctuations is captured by

$$E(k_z, t) = \langle \mathbf{v}, \mathbf{v} \rangle = E_u(k_z, t) + E_\psi(k_z, t),$$

where $E_u(k_z, t) = \langle u, u \rangle$ and $E_\psi(k_z, t) = \langle v, v \rangle + \langle w, w \rangle = -\langle \psi, \Delta \psi \rangle$. The angular brackets denote the standard $L_2[-1, 1]$ inner product, which induces the $L_2[-1, 1]$ norm

$$\|\mathbf{v}\|_2^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \int_{-1}^1 \mathbf{v}^*(y, k_z, t) \mathbf{v}(y, k_z, t) dy,$$

and the asterisk denotes the complex-conjugate transpose of vector \mathbf{v} . In view of the observation that \mathbf{v} does not depend on $\boldsymbol{\tau}_3$, we neglect Eq. (6c) in further analysis, yielding the following evolution model

$$\begin{bmatrix} \dot{\boldsymbol{\tau}}_1 \\ \dot{\boldsymbol{\tau}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & 0 \\ We \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{bmatrix}, \quad (7a)$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{C}_u \\ \mathbf{C}_v & 0 \\ \mathbf{C}_w & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{bmatrix}. \quad (7b)$$

The \mathbf{A} -operators determine the dynamical properties of system (7), and the \mathbf{C} -operators specify the static-in-time relations between velocity fluctuation components u , v , and w and polymer stress components $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$. These operators are We -independent and they are given by Eqs. (14) and (15) in Appendix A. The system of equations (7) is in a form suitable for the analysis carried out in Sec. III-B, where we provide an explicit characterization of the We -dependence for the transient growth of the velocity fluctuations.

B. Transient growth of kinetic energy density

By making use of basic results from linear systems theory, the lower block-triangular structure of the operator on the right-hand-side of Eq. (7a) can be exploited to formally determine the temporal evolution of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$

$$\begin{bmatrix} \boldsymbol{\tau}_1(t) \\ \boldsymbol{\tau}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11}(t) & 0 \\ We \mathbf{S}_{21}(t) & \mathbf{S}_{22}(t) \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau}_1(0) \\ \boldsymbol{\tau}_2(0) \end{bmatrix}. \quad (8)$$

Here, $\boldsymbol{\tau}_i(0)$ denotes the initial condition in $\boldsymbol{\tau}_i$, i.e. $\boldsymbol{\tau}_i(0) = \boldsymbol{\tau}_i(y, k_z, t=0)$, $i = 1, 2$. The operator $\mathbf{S}_{ii}(t)$ represents the solution to the following equation [21]

$$\dot{\mathbf{S}}_{ii}(t) = \mathbf{A}_{ii} \mathbf{S}_{ii}(t), \quad \mathbf{S}_{ii}(0) = \mathbf{I},$$

where \mathbf{I} is the identity operator, and,

$$\mathbf{S}_{21}(t) = \int_0^t \mathbf{S}_{22}(t-\xi) \mathbf{A}_{21} \mathbf{S}_{11}(\xi) d\xi.$$

For notational convenience the dependence on y , k_z , and β is suppressed in the above expressions. More precisely, at any fixed (We, β, k_z, t) , the \mathbf{S} -symbols in Eq. (8) denote operators that map initial values of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ (as a function of y) to the values of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ (as a function of y) at time t , i.e.

$$\begin{aligned} \boldsymbol{\tau}_1(y, k_z, t) &= [\mathbf{S}_{11}(k_z, t) \boldsymbol{\tau}_1(\cdot, k_z, 0)](y), \\ \boldsymbol{\tau}_2(y, k_z, t) &= We [\mathbf{S}_{21}(k_z, t) \boldsymbol{\tau}_1(\cdot, k_z, 0)](y) + [\mathbf{S}_{22}(k_z, t) \boldsymbol{\tau}_2(\cdot, k_z, 0)](y). \end{aligned} \quad (9)$$

It should be also noted that operators \mathbf{S}_{ij} are parameterized by β and that all of them are We -independent. By substituting these expressions for $\tau_1(t)$ and $\tau_2(t)$ into Eq. (7b) we finally arrive at

$$\begin{aligned} u(y, k_z, t) &= We [\mathbf{C}_u(k_z) \mathbf{S}_{21}(k_z, t) \tau_1(\cdot, k_z, 0)](y) + \\ &\quad [\mathbf{C}_u(k_z) \mathbf{S}_{22}(k_z, t) \tau_2(\cdot, k_z, 0)](y), \\ v(y, k_z, t) &= [\mathbf{C}_v(k_z) \mathbf{S}_{11}(k_z, t) \tau_1(\cdot, k_z, 0)](y), \\ w(y, k_z, t) &= [\mathbf{C}_w(k_z) \mathbf{S}_{11}(k_z, t) \tau_1(\cdot, k_z, 0)](y). \end{aligned}$$

Several conclusions can now be drawn about dynamics of velocity fluctuations in streamwise-constant inertialess flows of Oldroyd-B fluids without doing any detailed computations. First, the responses of wall-normal and spanwise velocity fluctuations are We -independent and they are caused by $\tau_1(0)$. Second, the streamwise velocity depends on both $\tau_1(0)$ and $\tau_2(0)$; the contribution of $\tau_2(0)$ to $u(t)$ is We -independent and the contribution of $\tau_1(0)$ to $u(t)$ scales linearly with the Weissenberg number. Third, operator \mathbf{A}_{21} in Eq. (7a) is essential to providing affine dependence of $u(t)$ on We ; this operator couples τ_1 to τ_2 and if it was zero the responses of all velocity components would be We -independent since We would be gone from Eqs. (7).

Since the presence of initial conditions in τ_1 introduces $\mathcal{O}(We)$ responses of the streamwise velocity fluctuation, we next examine the maximal transient growth of $u(t)$ (as a function of k_z , t , and β) arising from $\tau_1(0)$. For any fixed $(We, \beta; k_z, t)$, this quantity is determined by

$$\begin{aligned} G_{u1}(We, \beta; k_z, t) &= \sup_{\tau_1(0) \neq 0, \tau_2(0) \equiv 0} \frac{\|u(\cdot, k_z, t)\|_2^2}{\|\tau_1(\cdot, k_z, 0)\|_2^2} \\ &= \sup_{\|\tau_1(0)\|_2 = 1, \tau_2(0) \equiv 0} \|u(\cdot, k_z, t)\|_2^2 \\ &= We^2 \sigma_{\max}(\mathbf{C}_u(k_z) \mathbf{S}_{21}(\beta; k_z, t)) \\ &= We^2 \bar{G}_{u1}(\beta; k_z, t), \end{aligned} \quad (10)$$

where $\sigma_{\max}(\cdot)$ denotes the largest singular value of a given operator, and $\bar{G}_{u1}(\beta; k_z, t) = \sigma_{\max}(\mathbf{C}_u(k_z) \mathbf{S}_{21}(\beta; k_z, t))$ is a We -independent function. We will also pay attention to the contribution of different components of $\tau_1(0)$ to the transient growth of $u(t)$. For example, $\bar{G}_{u22}(\beta; k_z, t)$ will denote the maximal transient growth of $u(t)$ arising from the initial condition in τ_{22} (with all other initial conditions being set to zero) at $We = 1$; similar notation will be used to quantify the influence of the other two components of $\tau_1(0)$ on $u(t)$.

The function characterizing maximal transient growth of streamwise velocity fluctuations arising from the initial conditions in τ_1 (cf. Eq. (10)) in flows with $\beta = 0.1$, $\bar{G}_{u1}(0.1; k_z, t)$, is shown in Fig. 2; results for other values of β look similar and are not reported here for brevity. The finite-dimensional approximations of the wall-normal operators are obtained using the pseudospectral method [22]. All computations are performed in MATLAB with 50 Gauss-Lobatto points in the wall-normal direction; additional computations with a much larger number of grid points in y were used to confirm convergence. We observe similar trends in both Couette and Poiseuille flows with peak values of $\bar{G}_{u1}(0.1; k_z, t)$ occurring at $k_z = 0$ and $t \approx 0.25$. From the above discussion, it immediately follows that the largest contribution to the transient growth of u comes from the

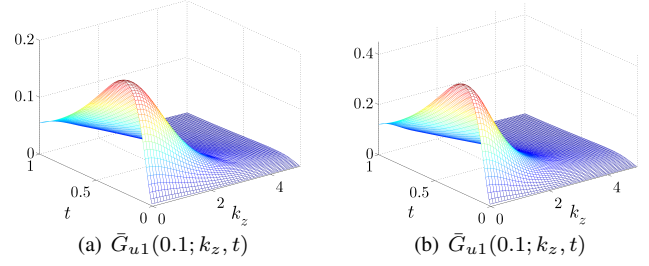


Fig. 2. Maximal transient growth of streamwise velocity fluctuations in (a) Couette and (b) Poiseuille flows with $\beta = 0.1$ arising from the initial condition in τ_1 . All other initial conditions have been set to zero.

initial conditions in τ_{22} ; this is because the maximal transient growth happens at $k_z = 0$ and *only* $\tau_{22}(0)$ contributes to $\bar{G}_{u1}(\beta; k_z = 0, t)$. This is further illustrated in Fig. 3, where maximal transient growth of streamwise velocity fluctuations caused by the different components of $\tau_1(0)$ in Couette flow with $\beta = 0.1$ is shown. The peak values in Fig. 3(a) are about three times larger than the peak values in Fig. 3(b), and about sixty times larger than the peak values in Fig. 3(c). This suggests that the initial conditions in τ_{22} create the largest transient growth of the streamwise velocity fluctuations, followed by the initial conditions in τ_{23} , followed by the initial conditions in τ_{33} . It is also noteworthy that the peaks of functions $\bar{G}_{u23}(0.1; k_z, t)$ and $\bar{G}_{u33}(0.1; k_z, t)$ occur at non-zero values of k_z ; on the other hand, similar to $\bar{G}_{u1}(0.1; k_z, t)$, function $\bar{G}_{u22}(0.1; k_z, t)$ achieves its maximum at $k_z = 0$.

Motivated by the observation that the largest transient growth of energy density for streamwise-constant velocity fluctuations takes place at $k_z = 0$, we next examine the linearized model with both $k_x = 0$ and $k_z = 0$. From the analysis of this model (not reported here) it follows that the streamwise velocity, $u(y, t)$, can be represented as $\{u(y, t) = We u_{22}(y, t) + e^{-t/\beta} u(y, 0); u_{22}(y, t) = -(e^{-t} - e^{-t/\beta}) [\partial_{yy}^{-1} (U''(y) + U'(y) \partial_y) \tau_{22}(\cdot, 0)](y)\}$. This indicates that the initial conditions in the streamwise velocity (or, equivalently, in τ_{12} ; at $k_x = k_z = 0$, $u(y, t) = -(1/\beta - 1) [\partial_{yy}^{-1} \partial_y \tau_{12}(\cdot, t)](y)$) create monotonically decaying We -independent responses of $u(y, t)$, with a rate of decay inversely proportional to the viscosity ratio β . On the other hand, even though $\tau_{22}(y, 0) \neq 0$ yields zero initial kinetic energy, the presence of initial conditions in τ_{22} generates streamwise velocity fluctuations, $We u_{22}(y, t)$, that scale linearly with the Weissenberg number and also exhibit temporal transient growth. We note that this feature arises solely from viscoelastic nature of the underlying fluid, and that the approach that allows for fluctuations in polymer stresses *but not* in velocities fails to identify it [18].

C. Comparison with inertial flows of Newtonian fluids

We next provide comparison of the above results with those for inertial flows of Newtonian fluids. By scaling length with the channel half height L , velocity with the largest base velocity U_o , and time with the diffusive time $\rho L^2/\eta_s$, the linearized evolution model for streamwise-constant fluctuations

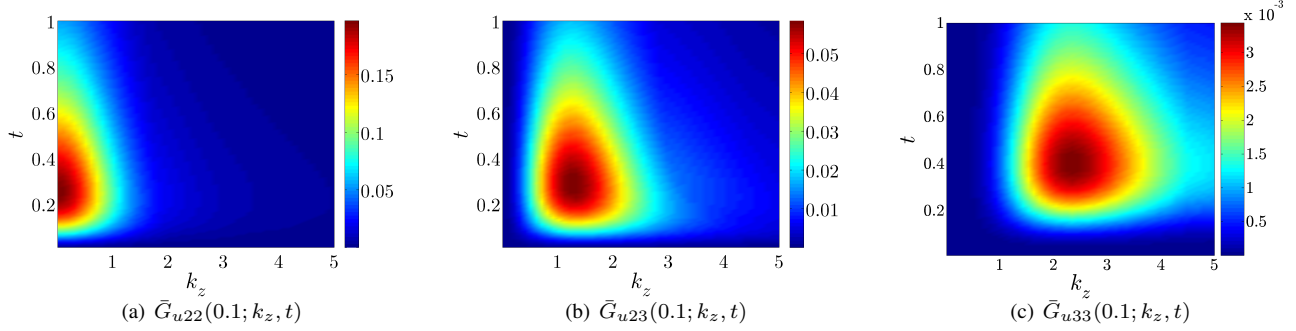


Fig. 3. Maximal transient growth of streamwise velocity fluctuations in Couette flow with $\beta = 0.1$ arising from the initial condition in: (a) τ_{22} ; (b) τ_{23} ; and (c) τ_{33} . All other initial conditions have been set to zero.

assumes the following form

$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{11} & 0 \\ Re \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{22} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \quad (11a)$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & \bar{\mathbf{C}}_u \\ \bar{\mathbf{C}}_v & 0 \\ \bar{\mathbf{C}}_w & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}. \quad (11b)$$

This model is obtained by eliminating pressure from the linearized NS equations and by expressing flow fluctuations in terms of the (y, z) -plane streamfunction $\phi_1 = \psi$ (cf. Sec. II-B) and the streamwise velocity $\phi_2 = u$. Here, $Re = \rho U_o L / \eta_s$ denotes the Reynolds number, and operators $\bar{\mathbf{A}}$ and $\bar{\mathbf{C}}$ are given by $\{\bar{\mathbf{A}}_{11} = \Delta^{-1} \Delta^2, \bar{\mathbf{A}}_{22} = \Delta, \bar{\mathbf{A}}_{21} = -ik_z U'(y); \bar{\mathbf{C}}_u = \mathbf{I}, \bar{\mathbf{C}}_v = ik_z, \bar{\mathbf{C}}_w = -\partial_y\}$, with Dirichlet boundary conditions on Δ , and Cauchy boundary conditions on Δ^2 . Note that $\bar{\mathbf{A}}_{11}$, $\bar{\mathbf{A}}_{22}$, and $\bar{\mathbf{A}}_{21}$, respectively, denote the Orr-Sommerfeld, Squire, and coupling operators in the streamwise-constant linearized NS equations with $Re = 1$.

Direct comparison of Eqs. (7) and (11) reveals a striking structural similarity between streamwise-constant inertialess flows of Oldroyd-B fluids and inertial flows of Newtonian fluids. In particular, these two equations can be represented graphically by the corresponding block diagrams in Figs. 4(a) and 4(b), respectively. The enabling mechanism for transient growth in Newtonian fluids is vortex tilting, which is embedded in operator $\bar{\mathbf{A}}_{21} = -ik_z U'(y)$. In the absence of vortex tilting, the responses of all velocity components are Re -independent and the dynamical properties of streamwise-constant flows of Newtonian fluids are governed by viscous dissipation. On the other hand, the key physical mechanism for transient growth in inertialess flows of Oldroyd-B fluids is polymer stretching, which is embedded in operator $\bar{\mathbf{A}}_{21} = \mathbf{F}_{21} + \mathbf{F}_{21\psi} \mathbf{C}_\psi$ (cf. Sec. II-B and Appendix A). We also observe that the Weissenberg number in viscoelastic fluids has a role similar to that of the Reynolds number in Newtonian fluids.

The above comparison suggests remarkable similarities between streamwise-constant inertialess flows of Oldroyd-B fluids and streamwise-constant inertial flows of Newtonian fluids. All these similarities are exhibited at the level of velocity fluctuation dynamics. Namely, as far as kinetic energy density is concerned, it is conceptually useful to think of inertialess flows of Oldroyd-B fluids in terms of inertial flows of Newtonian fluids. This analogy is made keeping in mind that polymer stretching and the Weissenberg number in elasticity-dominated flows of viscoelastic fluids effectively

take the role of vortex tilting and the Reynolds number in inertia-dominated flows of Newtonian fluids.

IV. CONCLUDING REMARKS

We have studied transient responses of velocity fluctuations in inertialess channel flows of viscoelastic fluids. By focusing on the analysis of streamwise-constant fluctuations, we are able to obtain a number of new analytical results. In contrast, most prior work on this topic has focused on the analysis of spanwise-constant fluctuations, which does not yield analytical results as readily and is prone to numerical difficulties in high-Weissenberg-number flows [16]. In addition, both velocity and polymer stress fluctuations may be non-zero in our work, which sets it apart from a recent paper that considered only non-zero (but 3D) polymer stress fluctuations in an upper convected Maxwell fluid [18].

The present work (i) makes clear that streamwise-constant velocity fluctuations in channel flows of viscoelastic fluids can undergo significant transient growth even in the absence of inertia; and (ii) reveals remarkable similarities between inertial flows of Newtonian fluids and inertialess flows of Oldroyd-B fluids.

APPENDIX

A. Dynamics of the streamwise-constant flow fluctuations

In this appendix we describe the equations governing evolution of infinitesimal streamwise-constant flow fluctuations in inertialess flows of an Oldroyd-B fluid. We also define the underlying operators in Eqs. (5), (6), and (7). By decomposing the velocity, pressure, and polymer stress fields into the sum of base and fluctuating parts (i.e., $\mathbf{V} = \bar{\mathbf{v}} + \mathbf{v}$, $P = \bar{P} + p$, $\mathbf{T} = \bar{\boldsymbol{\tau}} + \boldsymbol{\tau}$), and by neglecting nonlinear terms, Eqs. (2), (1b), and (1c) can be brought to the following form

$$0 = \beta \Delta \mathbf{v} + (1 - \beta) \nabla \cdot \boldsymbol{\tau} - \nabla p, \quad (12a)$$

$$0 = \nabla \cdot \mathbf{v}, \quad (12b)$$

$$\dot{\boldsymbol{\tau}} = \mathcal{L}(\boldsymbol{\tau}, \mathbf{v}), \quad (12c)$$

where $\mathcal{L}(\boldsymbol{\tau}, \mathbf{v})$ denotes linear flow fluctuation terms, i.e.

$$\begin{aligned} \mathcal{L}(\boldsymbol{\tau}, \mathbf{v}) = & \nabla \mathbf{v} + (\nabla \mathbf{v})^T - \boldsymbol{\tau} - We (\mathbf{v} \cdot \nabla \bar{\boldsymbol{\tau}} + \bar{\mathbf{v}} \cdot \nabla \boldsymbol{\tau}) + \\ & We (\boldsymbol{\tau} \cdot \nabla \bar{\mathbf{v}} + (\boldsymbol{\tau} \cdot \nabla \bar{\mathbf{v}})^T + \bar{\boldsymbol{\tau}} \cdot \nabla \mathbf{v} + (\bar{\boldsymbol{\tau}} \cdot \nabla \mathbf{v})^T). \end{aligned}$$

For purely harmonic fluctuations in z , e.g. $\mathbf{v}(y, z, t) = \Re(\mathbf{v}(y, k_z, t) e^{ik_z z})$, the linearized evolution model is given by Eqs. (5). Here, $\Re(\cdot)$ denotes the real part of a given

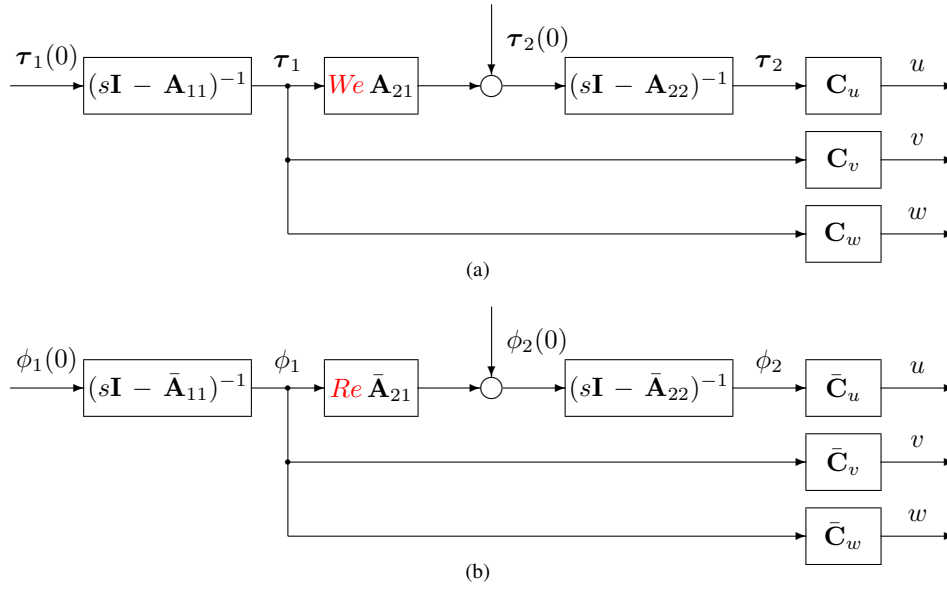


Fig. 4. The block diagrams of streamwise-constant (a) inertialess flows of Oldroyd-B fluids, cf. Eq. (7); and (b) inertial flows of Newtonian fluids, cf. Eq. (11). In Newtonian fluids transient growth comes from vortex tilting, i.e. operator $\bar{\mathbf{A}}_{21}$, and in viscoelastic fluids it comes from polymer stretching, i.e. operator \mathbf{A}_{21} . Note that, the Weissenberg number in inertialess flows of viscoelastic fluids assumes the role of the Reynolds number in inertial flows of Newtonian fluids.

quantity, and the \mathbf{F} -operators in Eqs. (5) are given by:

$$\begin{aligned} \mathbf{F}_{1\psi} &= \begin{bmatrix} 2ik_z\partial_y & -(\partial_{yy} + k_z^2) & -2ik_z\partial_y \end{bmatrix}^T, \\ \mathbf{F}_{2\psi} &= \begin{bmatrix} ik_z(U'(y)\partial_y - U''(y)) \\ -U'(y)\partial_{yy} \end{bmatrix}, \quad \mathbf{F}_{2u} = \begin{bmatrix} \partial_y \\ ik_z \end{bmatrix}, \\ \mathbf{F}_{21} &= \begin{bmatrix} U'(y) & 0 & 0 \\ 0 & U'(y) & 0 \end{bmatrix}, \quad \mathbf{F}_{3\psi} = -4ik_zU'(y)U''(y), \\ \mathbf{F}_{3u} &= 2U'(y)\partial_y, \quad \mathbf{F}_{32} = \begin{bmatrix} 2U'(y) & 0 \end{bmatrix}. \end{aligned} \quad (13)$$

Substitution of Eq. (3) into Eqs. (5) leads to the set of evolution equations (6) for the polymer stress components with the \mathbf{A} -operators given by

$$\begin{aligned} \mathbf{A}_{11} &= -\mathbf{I} + \mathbf{F}_{1\psi}\mathbf{C}_\psi, \quad \mathbf{A}_{22} = -\mathbf{I} + \mathbf{F}_{2u}\mathbf{C}_u, \\ \mathbf{A}_{21} &= \mathbf{F}_{21} + \mathbf{F}_{2\psi}\mathbf{C}_\psi, \quad \mathbf{A}_{31} = \mathbf{F}_{3\psi}\mathbf{C}_\psi, \\ \mathbf{A}_{32} &= \mathbf{F}_{32} + \mathbf{F}_{3u}\mathbf{C}_u. \end{aligned} \quad (14)$$

The \mathbf{C} -operators appearing in Eq. (7), which is convenient for quantifying the scaling of the kinetic energy density with the Weissenberg number, are given by

$$\begin{aligned} \mathbf{C}_u &= -(1/\beta - 1)\Delta^{-1} \begin{bmatrix} \partial_y & ik_z \end{bmatrix}, \\ \mathbf{C}_v &= ik_z\mathbf{C}_\psi, \quad \mathbf{C}_w = -\partial_y\mathbf{C}_\psi, \end{aligned} \quad (15)$$

where \mathbf{C}_ψ is defined in Eq. (4). The expressions for operators \mathbf{C}_v and \mathbf{C}_w are obtained by substituting $\psi = \mathbf{C}_\psi\tau_1$ (cf. Eq. (3)) into the equation relating the wall-normal and spanwise velocity fluctuations with the streamfunction, $\{v = ik_z\psi, w = -\partial_y\psi\}$.

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