CS 356 Unit 3

IEEE 754 Floating Point Representation

Floating Point

- Used to represent ____________ numbers (fractions) and ___________ numbers
  - Avogadro's Number: $+6.0247 * 10^{23}$
  - Planck's Constant: $+6.6254 * 10^{-27}$
  - Note: 32 or 64-bit integers can't represent this range
- Floating Point representation is used in HLL's like C by declaring variables as **`float`** or **`double`**

Fixed Point

- Unsigned and 2's complement fall under a category of representations called "_______________"
- The radix point is ___________ to be in a fixed location for all numbers [Note: we could represent fractions by implicitly assuming the binary point is at the left…A variable just stores bits…you can assume the binary point is anywhere you like]
  - Integers: 10011101. (binary point to right of LSB)
    - For 32-bits, unsigned range is 0 to ~4 billion
  - Fractions: .10011101 (binary point to left of MSB)
    - Range [0 to 1)
- **Main point**: By fixing the radix point, we _________ the range of numbers that can be represented
  - Floating point allows the radix point to be in a different location for each value

Bit storage

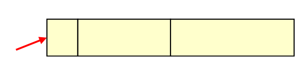
Fixed point Rep.

Floating Point Representation

CS:APP 2.4.2

- Similar to ______________used with decimal numbers
  - $\pm D.DDD * 10^{\pm exp}$
- Floating Point representation uses the following form
  - $\pm b.bbbb * 2^{\pm exp}$
  - 3 Fields: ________, ___________, ___________ (also called ____________ or significand)

Overall Sign of #



Normalized FP Numbers

- Decimal Example
  - $+0.754*10^{15}$ is _____ correct scientific notation
  - Must have exactly one _______________ before decimal point: ___________________
- In binary the only significant digit is _________
- Thus normalized FP format is:
- FP numbers will **always be** ______________ before being stored in memory or a reg.
  - The ______ is actually not stored but assumed since we always will store normalized numbers
  - If HW calculates a result of $0.001101*2^5$ it must normalize to $1.101000*2^2$ before storing

IEEE Floating Point Formats

- Single Precision (32-bit format)
  - ___ Sign bit (0=pos/1=neg)
  - ___ Exponent bits
    - _________ representation
    - More on next slides
  - ___ fraction (significand or mantissa) bits
  - Equiv. Decimal Range:
    - 7 digits x $10^{\pm 38}$

S	Exp.	Fraction

- Double Precision (64-bit format)
  - ___ Sign bit (0=pos/1=neg)
  - ___ Exponent bits
    - _________ representation
    - More on next slides
  - ___ fraction (significand or mantissa) bits
  - Equiv. Decimal Range:
    - 16 digits x $10^{\pm 308}$

S	Exp.	Fraction

Exponent Representation

- Exponent needs its own sign (+/-)
- Rather than using 2's comp. system we use Excess-N representation
  - Single-Precision uses Excess-127
  - Double-Precision uses Excess-1023
  - w-bit exponent => Excess-__________
  - This representation allows FP numbers to be easily compared
- Let E' = stored exponent code and E = true exponent value
- For single-precision: E' = E + 127
  - 2^1 => E = 1, E' = 128_{10} = 10000000_2
- For double-precision: E' = E + 1023
  - 2^{-2} => E = -2, E' = 1021_{10} = 01111111101_2

2's comp.	E' (stored Exp.)	Excess-127
-1	1111 1111	
-2	1111 1110	
-128	1000 0000	
+127	0111 1111	
+126	0111 1110	
+1	0000 0001	
0	0000 0000	

Comparison of 2's comp. & Excess-N

Q: Why don't we use Excess-N more to represent negative #'s

Comparison & The Format

- Why put the exponent field before the fraction?
  - Q: Which FP number is bigger: $0.9999*2^2$ or $1.0000*2^1$
  - A: We should look at the __________ first to compare FP values and only look at the _________ if the exponents are __________
- By placing the exponent field first we can compare entire FP values as single bit strings (i.e. as if they were ____________________)

0	10000010	0000001000
0	10000001	1110000000

010000010000000001000

010000000111110000000

< > = ???

Exponent Representation

- FP formats reserve the exponent values of all 1's and all 0's for special purposes
- Thus, for single-precision the range of exponents is -126 to + 127

E' (range of 8-bits shown)	E (=E'-127) and special values
255 = 11111111	
254 = 11111110	E'-127=+127
…	
128 = 10000000	E'-127=+1
127 = 01111111	E'-127=0
126 = 01111110	E'-127=-1
…	
1 = 00000001	E'-127=-126
0 = 00000000	

IEEE Exponent Special Values

Exp. Field	Fraction Field	Meaning

Single-Precision Examples

CS:APP 2.4.3

1

2^7=128 2^1=2

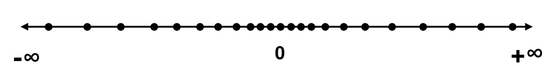
1	1000 0010	110 0110 0000 0000 0000 0000

2

+0.6875 = +0.1011

Floating Point vs. Fixed Point

- Single Precision (32-bits) Equivalent Decimal Range:
  - 7 significant decimal digits * $10^{\pm 38}$
  - Compare that to 32-bit signed integer where we can represent ±2 billion. How does a 32-bit float allow us to represent such a greater range?
  - FP allows for ________ but sacrifices ___________ (can't represent _______________ in its range)
- Double Precision (64-bits) Equivalent Decimal Range:
  - 16 significant decimal digits * $10^{\pm 308}$



12-bit "IEEE Short" Format

- 12-bit format defined just for this class (doesn't really exist)
  - 1 Sign Bit
  - 5 Exponent bits (using Excess-____)
    - Same reserved codes
  - 6 Fraction (significand) bits

1	5-bits	6-bits
S	E'	F
Sign Bit	*Exponent*	*Fraction*
0=pos. 1=neg.	*Excess-__*	*1.bbbbbb*

Examples

1

1	10100	101101

2 +21.75 = +10101.11

3

1	01101	100000

4 +3.625 = +11.101

ROUNDING

The Need To Round

CS:APP 2.4.4

- Integer to FP
  - +725 = 1011010101 = 1.011010101*2^9
    - If we only have 6 fraction bits, we ________ all fraction bits
- FP ADD / SUB

```
  5.9375 x 10^1            __________ x 10^5
+ 2.3256 x 10^5    ==>   + 2.3256     x 10^5
```

- FP MUL / DIV

```
        1.010110
*       1.110101
10.011101001110
```

```
         1.010110
       * 1.110101
          1010110
        1010110--
      1010110----
     1010110-----
  + 1010110------
  10.011101001110
```

Make sure to move the binary point

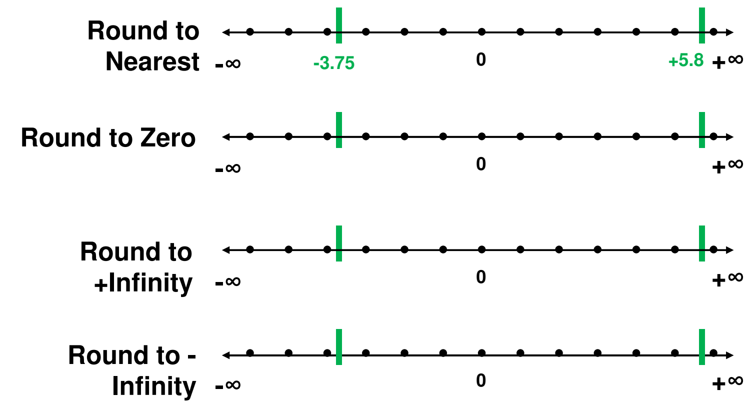
Rounding Methods

- 4 Methods of Rounding (you are only responsible for the first 2)

Round to _______ (Round to ______)	Normal rounding you learned in grade school. Round to the nearest representable number. If exactly halfway between, round to representable value w/ 0 in LSB (**i.e. nearest even fraction**).
Round towards __ (__________)	Round the representable value closest to but not greater in magnitude than the precise value. Equivalent to just dropping the extra bits.
Round toward ___ (Round Up)	Round to the closest representable value greater than the number
Round toward ___ (Round Down)	Round to the closest representable value less than the number

Number Line View Of Rounding Methods

Green lines are FP results that fall between two representable values (dots) and thus need to be rounded



Rounding to Nearest Method

- Same idea as rounding in decimal
- Examples: Round 1.23xx to the nearest 1/100th
  - 1.2351 to 1.2399 => round ____________
  - 1.2301 to 1.2349 => round ____________
  - 1.2350 => Rounding options 1.23 or 1.24
    - Choose the option with an ______ digit in the LS place (i.e. _____)
  - 1.2450 => Rounding options 1.24 or 1.25
    - Choose the option with an ______ digit in the LS place (i.e. _____)
- Which option has the even digit is essentially a _______ probability of leading to rounding up vs. rounding down
  - Attempt to reduce ________ in a sequence of operations

Rounding in Binary

- What does "exactly" half way correspond to in binary (i.e. 0.5 dec. = ??)
- Hardware will keep some __________ bits beyond what can be stored to help with rounding
  - Referred to as the _______ bit(s), _______ bit, and __________ bit (GRS)
- Thus, if the additional bits are:
  - 10…0 = _____________
  - 0x…x = ________ half way (round ______)
  - Anything else = _________ half way (round __)

0.5 = __ __ __ __

Bits that fit in FRAC field

$1.\underbrace{010010}\mathbf{101} \times 2^4$

Additional bits: 101

Round to Nearest

1.001100110 x 2^4 — *Additional bits: 110*

1.111111101 x 2^4 — *Additional bits: 101*

1.001101001 x 2^4 — *Additional bits: 001*

Round to Nearest

- In all these cases, the numbers are halfway between the 2 possible round values
- Thus, we round to the value w/ 0 in the LSB

1.001100100 x 2^4 — *Additional bits: 100*

1.111111100 x 2^4 — *Additional bits: 100*

1.001101100 x 2^4 — *Additional bits: 100*

0		

Round to 0 (Chopping)

- Simply drop the G,R,S bits and take fraction as is

1.001100001 x 2^4 (GRS: 001)

1.001101101 x 2^4 (GRS: 101)

1.001100111 x 2^4 (GRS: 111)

0	10011	

0	10011	

0	10011	

MAJOR IMPLICATIONS FOR PROGRAMMERS

FP Addition/Subtraction

CS:APP 2.4.5

- FP addition/subtraction is NOT _______________
  - Because of _____________ and use of _____________
    (a+b)+c ≠ a+(b+c)
  - Add similar, small magnitude numbers before larger magnitude numbers
- Example of rounding

  (0.0001 + 98475) – 98474 ≠ 0.0001 + (98475-98474)

  ____

  ____
- Example of infinity

  $1 \quad + \quad 1.11\ldots1*2^{127} \quad - \quad 1.11\ldots1*2^{127}$

Floating point MUL/DIV

- Also not ____________________
- Doesn't ________________ over addition
  - a*(b+c) ≠ a*b + a*c
  - Example†:
    - (big1 * big2) / (big3 * big4) => ____________________
    - 1/big3 * 1/big4 * big1 * big2 => ____________________
    - (big1 / big3) * (big2 / big4) => ____________________
- Note: Take care even with integer mul/div
  - F = (9/5)*C + 32
  - Should be F = __________________

†https://www.soa.org/News-and-Publications/Newsletters/Compact/2014/may/Losing-My-Precision--Tips-For-Handling-Tricky-Floating-Point-Arithmetic.aspx

FP Comparison

- Beware of equality (==) check or even less- or greater-than
- Generally don't use FP as _________ counters
- Common approach to replace equality check
  - Check if ___________ of two values is within some _________
  - Many questions are raised by this…(what epsilon, what about sign, transitive equality)?

```
float x = 0.2 + 0.3; // 0.5?
float y = 0.15 + 0.35; // 0.5?
if(x == y) printf("Equal\n");
```

Will "Equal" be printed?

```
double t;
int cnt=0;
for(t=0.0; t < 1.0; t += 0.1)
{
  printf("%d\n", cnt++);
}
```

What values of 'cnt' will be printed?

```
bool simple_within(
  float a, float b, float eps)
{
  return fabs(a-b) < eps;
}
```


FP & Compiler Optimizations

- Suppose we want to compute:

  x = a + b + c;

  y = b + c + d;
- Can the compiler optimize this as:

  temp = b + c;

  x = a + temp;

  y = temp + d;

Floating point values in C

CS:APP 2.4.6

- Two types: `float` and `double`
  - IEEE floating point when supported
  - Rounds to even
- No standard way to __________ rounding
- No standard way to get __________ values

Casting and C

Cast	Overflow Possible?	Rounding Possible?	Notes
int to float			
int to double			
float to double			
double to float			
float/double to int			Round to 0 is used to truncate fractional values (i.e. 1.9 => 1) If overflow, use _________ int.

FURTHER INQUIRY

Rounding Implementation

- There may be a large number of bits after the fraction
- To implement any of the methods we can keep only a subset of the extra bits after the fraction [hardware is finite]
  - **G**uard bits: bits immediately after LSB of fraction (many HW implementations keep up to 16 additional guard bits)
    - ****Lookup online the usage & importance of these guard bits****
  - **R**ound bit: bit to the right of the guard bits
  - **S**ticky bit: Logical OR of all other bits after Guard & R bits

`1.01001010010 x 2^4`

Logical OR (output is '1' if any input is '1', '0' otherwise

`1.010010101 x 2^4`

GRS

We can perform rounding to a 6-bit fraction using just these 3 bits.

More

- Some links
  - https://docs.oracle.com/cd/E19957-01/806-3568/ncg_goldberg.html
  - http://floating-point-gui.de/