# Unit 2

Integer Operations

(Arithmetic, Overflow, Bitwise Logic, Shifting)

## Skills & Outcomes

- You should know and be able to apply the following skills with confidence
  - Perform addition & subtraction in unsigned & 2's complement system
  - Determine if overflow has occurred
  - Perform bitwise operations on numbers
  - Perform logic and arithmetic shifts and understand how they can be used for multiplication/division
  - Understand arithmetic in binary and hex

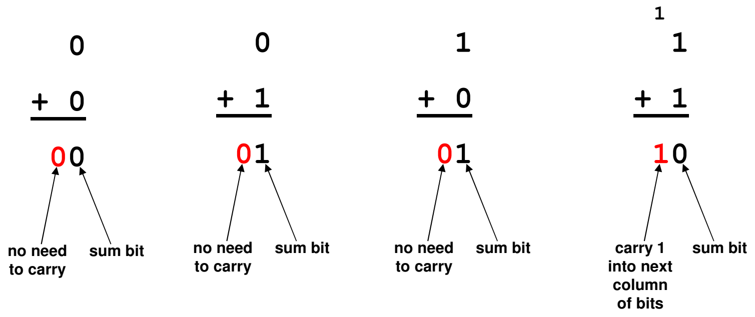
## UNSIGNED BINARY ARITHMETIC

## Binary Arithmetic

- Can perform all arithmetic operations (+,-,*,÷) on binary numbers
- Can use same methods as in decimal
  - Still use carries and borrows, etc.
  - Only now we carry when sum is __ or more rather than 10 or more (decimal)
  - We borrow ___'s not 10's from other columns
- Easiest method is to add bits in your head in decimal (1+1 = 2) then convert the answer to binary ($2_{10} = 10_2$)

# Binary Addition

- In decimal addition we carry when the sum is 10 or more
- In binary addition we carry when the sum is 2 or more
- Add bits in binary to produce a sum bit and a carry bit



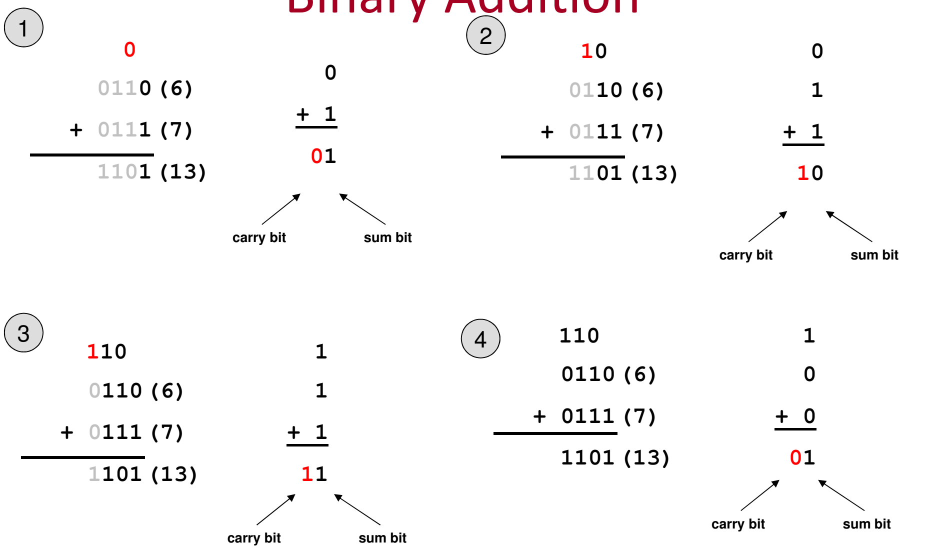
# Binary Addition & Subtraction

```
  0 1 1 1 (7)          1 0 1 0 (10)
+ 0 0 1 1 (3)        – 0 1 0 1  (5)
```

# Binary Addition

```
  110
  0110 (6)
  8 4 2 1
+ 0111 (7)
  1101 (13)
```

# Binary Addition



# Hexadecimal Arithmetic

- Same style of operations
  - Carry when sum is 16 or more, etc.

```
  4 D16          __ __
                 16  1
+ B 516          __ __
-------          16  1
```

# Binary Multiplication

- Like decimal multiplication, find each partial product and __________ them, then sum them up
- **Multiplying two *n*-bit numbers yields at most a ______-bit product**

```
      0 1 1 0 (6)
  *   0 1 0 1 (5)
  -----------
                  }  Partial Products



+ ---------------
                  <-- Sum of the partial products
```

# Binary Division

- Use the same long division techniques as in decimal
- **Dividing two *n*-bit numbers may yield an n-bit quotient and n-bit remainder**

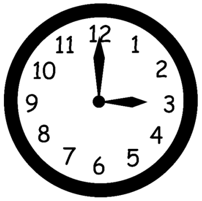
```
                0 1 0 1 r.1  (5 r.1)10
(2)10     10 | 1 0 1 1        (11)10
              -1 0
              ----
                 0 1
                -0 0
                ----
                   1 1
                  -1 0
                  ----
                   0 1
```

"Taking the 2's complement"

## SUBTRACTION THE EASY WAY

# Modulo Arithmetic

- The primary difference between how humans and computers perform arithmetic is the finite ________________ of computers
  - As humans we can use more digits (precision) as needed
  - Computers can only used a _________ set of bits
    - Much like the odometer on your car once you go too many miles the values will wrap from 999999 to 000000
    - Essentially all computer arithmetic is ___________ arithmetic
    - If we have a width of w bits, then all operations are module ______
- This leads to alternate approaches to arithmetic
  - Example: Consider how you could change the clock time from 5 p.m. to 3 p.m. if you can't ________ hours



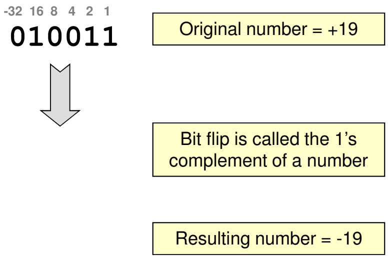
# Taking the Negative

CS:APP 2.3.3

- **Question**: Given a number in 2's complement how do we find its negative (i.e. -1 * X)
- **Answer**: By "______________________________"
  - 0110 = +6 => -6 = 1010
  - Operation defined as:
    1. ______________________________
    2. ______________________________
       (i.e. finish with the same # of bits as we start with)
  - See next slides for example

# Taking the 2's Complement

- Invert (flip) each bit (take the 1's complement)
  - 1's become 0's
  - 0's become 1's
- Add 1 (drop final carry-out, if any)



Original number = +19

Bit flip is called the 1's complement of a number

Resulting number = -19

Important: Taking the 2's complement is equivalent to taking the negative (negating)

# Taking the 2's Complement

1

-32 16 8 4 2 1

101010

Original number = -22

Take the 2's complement yields the negative of a number

Resulting number = +22

Taking the 2's complement again yields the original number (the operation is symmetric)

Back to original = -22

2

0000

Original # = 0

Take the 2's complement

2's comp. of 0 is __

3

1000

Original # = -8

Take the 2's complement

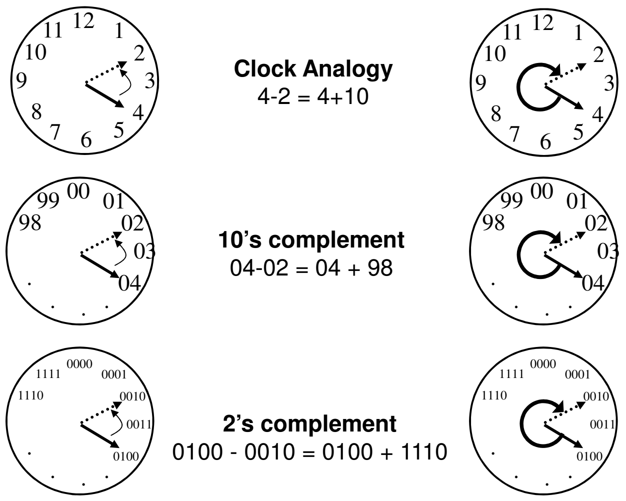
Negative of -8 is ___

(i.e. no positive equivalent, but this is not a huge problem)

The same algorithms regardless of unsigned or signed

# ADDITION AND SUBTRACTION

## Radix Complement



**Clock Analogy**
4-2 = 4+10

**10's complement**
04-02 = 04 + 98

**2's complement**
0100 - 0010 = 0100 + 1110

When using **modulo arithmetic**, **subtraction** can always be converted to **addition.**

## 2's Complement Addition/Subtraction

CS:APP 2.3.1
CS:APP 2.3.2

- Addition
  - Sign of the numbers _______________
  - Add column by column
  - Drop any final _______________
    - The secret to modulo arithmetic
- Subtraction
  - Any subtraction (A-B) can be converted to addition (_______) by taking the _______________of B
  - (A-B) becomes (A + __________ )
  - Drop any carry-out
    - The secret to modulo arithmetic

## 2's Complement Addition

- No matter the sign of the operands just add as normal
- Drop any extra carry out

```
    0011 (3)          1101 (–3)
  + 0010 (2)        + 0010  (2)
  ----------        ----------

    0011  (3)         1101 (–3)
  + 1110 (–2)       + 1110 (–2)
  ----------        ----------
```

# Unsigned and Signed Addition

- Addition process is the __________ for both unsigned and signed numbers
  - – Add columns right to left
- Examples:

```
            If unsigned  If signed
   1001
 + 0011
```

# 2’s Complement Subtraction

- Take the 2’s complement of the subtrahend (bottom #) and add to the original minuend (top #)
- Drop any extra carry out

```
   0011 (+3)            1101 (–3)
 – 0010 (+2)          – 1110 (–2)
```

# Unsigned and Signed Subtraction

- Subtraction process is the same for both unsigned and signed numbers
  - – Convert A – B     to     A + Comp. of B
  - – Drop any final carry out
- Examples:

```
         If unsigned  If signed
   1100     (12)        (–4)
 – 0010      (2)         (2)      ⇨
```

```
                                  If unsigned   If signed
```

# Important Note

- Almost all computers use 2's complement because…
- The same addition and subtraction ____________ can be used on unsigned and 2's complement (signed) numbers
- Thus we only need one set of ________________________________ to perform operations on both unsigned and signed numbers

# OVERFLOW

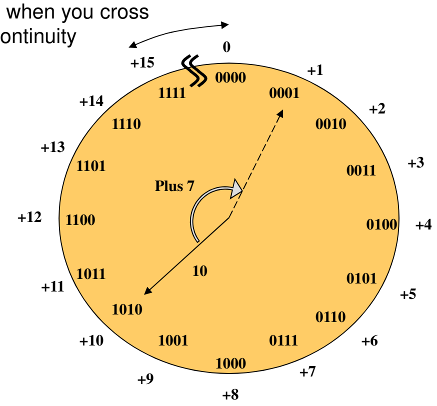
# Overflow

CS:APP 2.3.1
CS:APP 2.3.2

- Overflow occurs when the result of an arithmetic operation is ______________ ___________________________________
- Conditions and tests to determine overflow depend on the ________ being used
  - Different algorithms for detecting overflow based on ____________________

# Unsigned Overflow

Overflow occurs when you cross this discontinuity



**10 + 7 = 17**

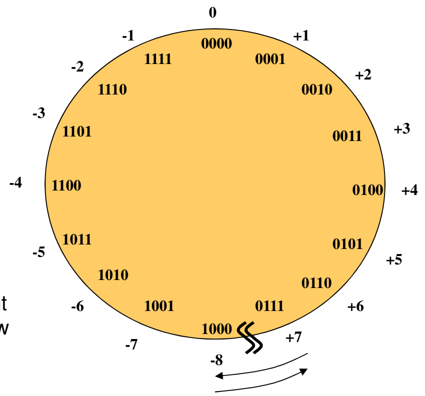
With 4-bit *unsigned* numbers we can only represent 0 – 15. Thus, we say overflow has occurred.

# 2's Complement Overflow

5 + 7 = +12

-6 + -4 = -10

With 4-bit *2's complement* numbers we can only represent -8 to +7. Thus, we say overflow has occurred.



Overflow occurs when you cross this discontinuity

# Overflow in Addition

- Overflow occurs when the result of the addition cannot be represented with the given number of bits.
- Tests for overflow:
  - Unsigned: if __________ [result __________ than inputs]
  - Signed: if ______________ [result has inappropriate sign]

```
 1 1     If unsigned   If signed
  1101      (13)         (-3)
+ 0100       (4)          (4)
  0001      (17)         (+1)
          Overflow    No Overflow
          Cout = 1       n + p
```

```
 0 1     If unsigned   If signed
  0110       (6)          (6)
+ 0101       (5)          (5)
  1011      (11)         (-5)
         No Overflow    Overflow
          Cout = 0     p + p = n
```

# Overflow in Subtraction

- Overflow occurs when the result of the subtraction cannot be represented with the given number of bits.
- Tests for overflow:
  - Unsigned: if _______ [expect negative result]
  - Signed: ________________[result has inappropriate sign]

```
        If unsigned   If signed
  0111      (7)          (7)
- 1000      (8)         (-8)
           (-1)         (15)

             Desired
             Results
```

⇨

```
  0111_
  0111  A
  0111  1's comp. of B
+    1  Add 1
  1111  (15)        (-1)
     If unsigned  If signed
      Overflow    Overflow
      Cout = 0    p + p = n
```

# MULTIPLICATION AND DIVISION

# Binary Multiplication

CS:APP 2.3.4

- **Multiplying two *n*-bit numbers yields at most a *2*n*-bit product**
- Multiplication operations on a modern processor can take ______ **times** longer than addition operations

```
          0 1 1 0 (6)

      *   0 1 0 1 (5)
      -----------
          0 1 1 0   ┐
        0 0 0 0     │
      0 1 1 0       ├ Partial Products
  + 0 0 0 0         ┘
  ---------------
    0 0 1 1 1 1 0  ← Sum of the partial products
```

# Binary Division

- **Dividing two *n*-bit numbers may yield an n-bit quotient and n-bit remainder**
- Division operations on a modern processor can take ________ **times** longer than addition operations

```
                   0 1 0 1 r.1  (5 r.1)_10
(2)_10       10 | 1 0 1 1       (11)_10
                -1 0 ↓    |
                   0 1    |
                  -0 0    ↓
                     1 1
                    -1 0
                     0 1
```

# Unsigned Multiplication Review

- Same rules as decimal multiplication
- Multiply each bit of Q by M shifting as you go
- An m-bit * n-bit mult. produces an m+n bit result
- Notice each partial product is a shifted copy of M or 0 (zero)

```
      1010  M (Multiplicand)
   *  1011  Q (Multiplier)
      1010
     1010_  PP(Partial
    0000__      Products)
+  1010___
  01101110  P (Product)
```

# Signed Multiplication Techniques

- When multiplying signed (2's comp.) numbers, some new issues arise
- Must sign extend partial products (out to 2n bits)

**Without Sign Extension… Wrong Answer!**

```
      1001  = -7
   *  0110  = +6
      0000
     1001_
    1001__
+  0000___
  00110110  = +54
```

**With Sign Extension… Correct Answer!**

```
      1001  = -7
    * 0110  = +6
  00000000
  1111001_
  111001__
+ 00000___
  11010110  = -42
```

# Signed Multiplication Techniques

- Also, must worry about negative multiplier
  - MSB of multiplier has negative weight
  - If MSB=1, multiply by -1 (i.e. take 2's comp. of multiplicand)

**With Sign Extension but w/o consideration of MSB… Wrong Answer!**

```
      1100  = -4
    * 1010  = -6
  00000000
  1111100_
  000000__
+ 11100___
  11011000  = -40
```

**With Sign Extension and w/ consideration of MSB… Correct Answer!**

Place Value: -8
Multiply by -1

```
      1100  = -4
    * 1010  = -6
  00000000
  1111100_
  000000__
+ 00100___
  00011000  = +24
```

**Main Point: Signed and Unsigned Multiplication require different techniques…Thus different instructions.**

# BITWISE & LOGIC OPERATIONS

## Modifying Individual Bits

CS:APP 2.1.7

- Suppose we want to change only a single bit (or a few bits) in a variable [i.e. `char v;`] _________________ the other bits
  - Set the LSB of v to 1 w/o affecting other bits
    - Would this work? **`v = 1;`**
  - Set the upper 4 bits of v to 1111 w/o affecting other bits
    - Would this work? **`v = 0xf0;`**
  - Clear the lower 2 bits of v to 00 w/o affecting other bits
    - Would this work? **`v = 0;`**
  - No!!! Assignment changes ALL bits in a variable
- **Because the smallest unit of data in computers is usually a ___________, manipulating individual bits requires us to use BITWISE OPERATIONS.**
  - **AND = &**
  - **OR = |**
  - **XOR = ^**
  - **NOT = ~**

| Bit: | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|
| | ? | ? | ? | ? | ? | ? | ? | ? |

**1-byte variable**

| ? | ? | ? | ? | ? | ? | ? | 1 |
|---|---|---|---|---|---|---|---|

**Desired v**

**(change LSB to 1)**

| 1 | 1 | 1 | 1 | ? | ? | ? | ? |
|---|---|---|---|---|---|---|---|

**Desired v**

**(change upper 4 bits to 1111)**

| ? | ? | ? | ? | ? | ? | 0 | 0 |
|---|---|---|---|---|---|---|---|

**Desired v**

**(change lower 2 bits to 00)**

## Using Bitwise Ops to Change Bits

- **ANDs** can be used to **clear a bit** (make it '0') or leave it unchanged
- **ORs** can be used to **set a bit** (make it '1') or leave it unchanged
- **XORs** can be used to **invert a bit** (flip it) or leave it unchanged

| X | Y | AND | |
|---|---|---|---|
| 0 | 0 | 0 | Force '0' |
| 0 | 1 | 0 | |
| 1 | 0 | 0 | Pass |
| 1 | 1 | 1 | |

0 AND y = __
1 AND y = __
y AND y = y

| X | Y | OR | |
|---|---|---|---|
| 0 | 0 | 0 | Pass |
| 0 | 1 | 1 | |
| 1 | 0 | 1 | Force '1' |
| 1 | 1 | 1 | |

0 OR y = __
1 OR y = __
y OR y = 1

| X | Y | XOR | |
|---|---|---|---|
| 0 | 0 | 0 | Pass |
| 0 | 1 | 1 | |
| 1 | 0 | 1 | Invert |
| 1 | 1 | 0 | |

0 XOR y = __
1 XOR y = NOT __
y XOR y = 0

| | | |
|---|---|---|
| **Identity** | **0 OR Y = __** | **1 AND Y = __** |
| **Null Ops** | **1 OR Y = __** | **0 AND Y = __** |
| **Idempotency** | **Y OR Y = Y** | **Y AND Y = Y** |

## Bitwise Operations

CS:APP 2.1.7

- The C AND , OR, XOR, NOT bitwise operations perform the operation on each pair of bits of 2 numbers

```
     0xa5   →     1010 0101
AND  0xf0       & 1111 0000
            ←

     0xa5   →     1010 0101
 OR  0xf0       | 1111 0000
            ←

     0xa5   →     1010 0101
XOR  0xf0       ^ 1111 0000
            ←

NOT  0xa5   →   ~ 1010 0101
            ←
```

```
#include <stdio.h> // C-Library
                   // for printf()

int main()
{
  char a = 0xa5;
  char b = 0xf0;

  printf("a & b = %x\n", a & b);
  printf("a | b = %x\n", a | b);
  printf("a ^ b = %x\n", a ^ b);
  printf("~a = %x\n",    ~a);
  return 0;
}
```

**C bitwise operators:**

```
& = AND
| = OR
^ = XOR
~ = NOT
```

# Logical vs. Bitwise Operations

CS:APP 2.1.8

- The C language has two types of logic operations
  - Logical and Bitwise
- Logical Operators (_______________)
  - Interpret entire value as either _____ (non-zero) or _____ (zero)
- Bitwise Operators (_______________)
  - Applies the logical operation on each __________ of the inputs

```
#include <stdio.h>
int main()
{
  int x = 1, y = 2;
  int z1 = x && y;
  int z2 = x & y;
  printf("z1=%d, z2=%d\n",z1,z2);

  char x = 1;
  if( !x ) { printf("L1\n"); }
  if( ~x ) { printf("L2\n"); }
  return 0;
}
```

```
   0000 0001=T
&& 0000 0010=T
   0000 0001=T
```

```
    0000 0001
  & 0000 0010
F = 0000 0000
```

```
! 0000 0001=T
  0000 0000=F
```

```
  ~ 0000 0001
T = 1111 1110
```

```
!! 0101 0111=T
   0000 0001=T
```

**Important Note**: Since !(non-zero) = 0; and !0 = 1
So !!35=1. And !!-109=1

# Application: Swapping via XORs

- Swapping variables can be done with a 3rd 'temp' variable
- For bitwise swapping, XORs can be used

```
#include <stdio.h>
int main()
{
  int x = 0x59, y = 0xd3;
  int temp = x;
  x = y;
  y = temp;

  return 0;
}
```

**Traditional swap with 'temp'**

**XOR swap**

```
#include <stdio.h>
int main()
{
  int x = 0x59, y = 0xd3;




  return 0;
}
```

```
0101 1001=x
```

```
1101 0011=y
```

# Exercises

- Determine if an integer is odd (w/o % operator).

```
bool isOdd(int x)
{

}
```

- Determine if an integer is a multiple of 4 (w/o % operator).

```
bool isMultOf4(int x)
{

}
```

Arithmetic and Logical Shifts

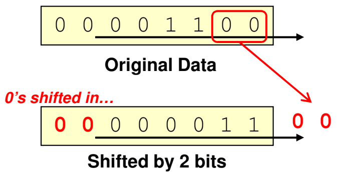
# SHIFT OPERATIONS

# Shift Operations

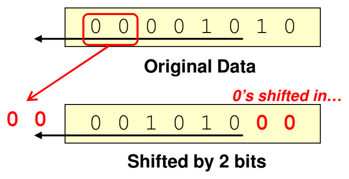
CS:APP 2.1.9

- Shifts data bits either left or right
  - Bits shifted out and _______________ on one side
  - Usually (but not always) 0's are shifted in on the other side
- Shifting is equivalent to multiplying or dividing by powers of __
- 2 kinds of shifts
  - Logical shifts (used for _______________ numbers)
  - Arithmetic shifts (used for _____________ numbers)

*Right* Shift by 2 bits:

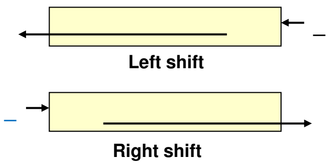


*Left* Shift by 2 bits:

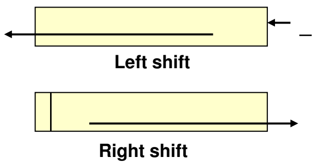


# Logical Shift vs. Arithmetic Shift

- Logical Shift
  - Use for __________ or non-numeric data
  - Will always shift in ___'s whether it be a left or right shift

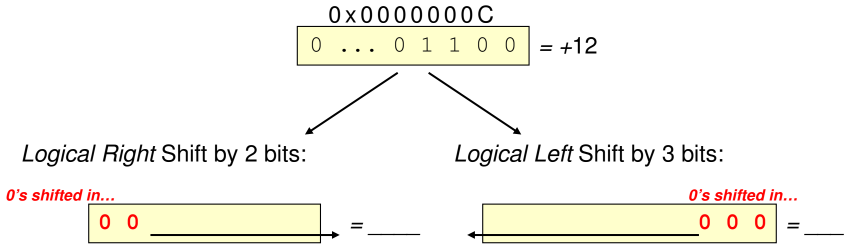


- Arithmetic Shift
  - Use for ___________ data
  - Left shift will shift in 0's
  - Right shift will sign extend (__________ the sign bit) rather than shift in 0's
    - If negative number...stays ________ by shifting in __'s
    - If positive...stays _________ by shifting in ___'s



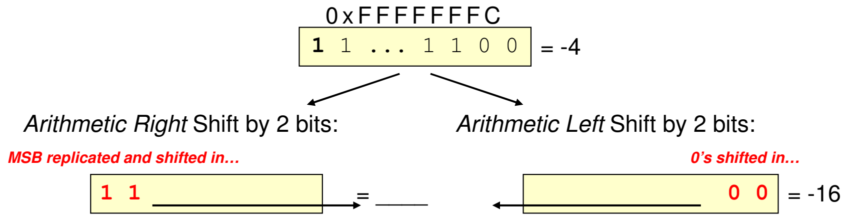
# Logical Shift

- 0's shifted in
- Only use for operations on *unsigned* data
  - Right shift by n-bits = ____________ by $2^n$
  - Left shift by n-bits = ____________ by $2^n$



# Arithmetic Shift

- Use for operations on *signed* data
- Arithmetic Right Shift – replicate MSB
  - Right shift by n-bits = Dividing by $2^n$
- Arithmetic Left Shift – shifts in 0's
  - Left shift by n-bits = Multiplying by $2^n$


0xFFFFFFFC
1 1 ... 1 1 0 0
= -4
Arithmetic Right Shift by 2 bits:
MSB replicated and shifted in...
1 1
= ____
Arithmetic Left Shift by 2 bits:
0's shifted in...
0 0
= -16

***Notice if we shifted in 0's (like a logical right shift) our result would be a positive number and the division wouldn't work***

***Notice there is no difference between an arithmetic and logical left shift. We always shift in 0's.***

# Multiplying by Non-Powers of 2

CS:APP 2.3.6

- Left shifting by n-bits allow us to multiply by $2^n$
- But what if I have to multiply a number by a *non-power* of 2 (i.e. 17*x). Can we still use shifting?
  - ________. Break constant into a _____ using ____________ coefficients
  - 17x = _________________
- Exercise: How many adds/shift would be needed to compute 14*x
  - ____________________________ OR
  - ____________________________

17=

| _ | _ | _ | _ | _ |
|---|---|---|---|---|
| 16 | 8 | 4 | 2 | 1 |

```
int mul17(int x)
{
  return 17*x;
}
```

Written Code

```
sall    $4, %edx
addl    %edx, %eax
```

```
int mul17(int x)
{
  int x16 = ________;
  return __________;
}
```

Optimized Assembly
(Equivalent C)

Compiler will determine when ___________ become ________ than constant multiplication

# Integer Division By Shifting

CS:APP 2.3.7

- What is 5/2?
  - ______
- Is 5/2 = (5 >> 1)
  - ________

5 =

| 0 | 1 | 0 | 1 |
|---|---|---|---|
| -8 | 4 | 2 | 1 |

5>>1 =

| | | | | _ |
|---|---|---|---|---|
| -8 | 4 | 2 | 1 | 0.5 |

- What is -5/2?
  - ______
- Is -5/2 = (-5 >> 1)
  - ______

-5 =

| | | | |
|---|---|---|---|
| -8 | 4 | 2 | 1 |

-5>>1 =

| | | | | _ |
|---|---|---|---|---|
| -8 | 4 | 2 | 1 | 0.5 |

**Main Point**: Rounding ______ when using shifting to divide a ________ number.

# Dividing Negative Numbers

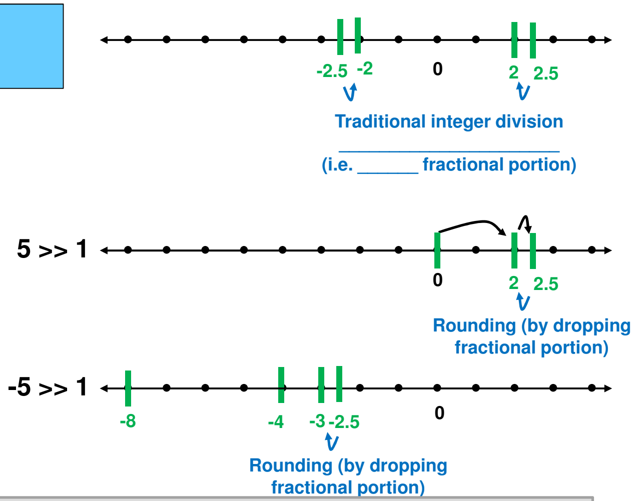
Traditional integer rounding

+5>>1

| 0 | 0 | 1 | 0 | 1 |
|---|---|---|---|---|
| -8 | 4 | 2 | 1 | 0.5 |

-5>>1

| 1 | 1 | 0 | 1 | 1 |
|---|---|---|---|---|
| -8 | 4 | 2 | 1 | 0.5 |


-2.5
-2
0
2
2.5
Traditional integer division
(i.e. ______ fractional portion)
5 >> 1
0
2
2.5
Rounding (by dropping fractional portion)
-5 >> 1
-8
-4
-3
-2.5
0
Rounding (by dropping fractional portion)

**Main Point**: Dividing numbers in the 2's complement system causes rounding to the _________________________, not toward _____ as desired.

# Biasing

- Summary: Dividing x / $2^k$ by performing (x >> k)…
  - Works when ______ OR when ______ & x is a multiple of __
  - Doesn't work when _____and x is NOT a multiple of ____
- Idea to solve the problem:
  - ______ some value (aka a ___ value) to x before _______ that will correct for the rounding issue
  - Add ________ (i.e. _______)

```
 -4  = 1 1 0 0
-4>>1 = 1 1 1 0  -2

 -5  = 1 0 1 1
-5>>1 = 1 1 0 1  -3

 -5    1 0 1 1


-4>>1 = 1 1 1 0  -2
```

# More Examples

- -8 / 4 = (-8 >> 2)
  - Bias by ________
  - (-8 + ___) >> 2

```
   -8 = 1 0 0 0
-8>>2 = 1 1 1 0   -2
```

- -7 / 4 = (-7 >> 2)
  - Bias by ________
  - (-7 + ___) >> 2

-2

```
   -7 = 1 0 0 1
-7>>2 = 1 1 0 0   -2
```

- -20 / 16 = (-20 >> 4)
  - Bias by _______
  - (-20 + ____) >> 4

-1

# CS:APP Practice 2.43 (tweaked)

```
#define M /* mystery number 1 */
#define N /* mystery number 2 */

int arith(int x, int y)
{
  int result = x*M + y/N;
  return result;
}

/* Translation of assembled code for
   a given value of M and N */
int optarith(int x, int y)
{
  int t = x;
  x <<= 5;
  x -= t;
  if(y < 0) y += 3;
  y >>= 2;
  return x + y;
}
```

**What were M and N when the code was compiled?**